

b Now $V^2 = \pi^2 x^4 (100 - 4x^2)$
 $= \pi^2 (100x^4 - 4x^6)$

$$\therefore \frac{d(V^2)}{dx} = \pi^2 (400x^3 - 24x^5)$$
 $= 8\pi^2 x^3 (50 - 3x^2)$
 $= 8\pi^2 x^3 (\sqrt{50} + \sqrt{3}x)(\sqrt{50} - \sqrt{3}x)$
 $\therefore \frac{d(V^2)}{dx} = 0 \text{ when } x = \sqrt{\frac{50}{3}} \text{ (as } x > 0\text{)}$

and $\frac{d(V^2)}{dx}$ has sign diagram:

∴ the maximum volume occurs when $x = \sqrt{\frac{50}{3}} \approx 4.08$

∴ radius $\approx 4.08 \text{ cm}$, height $= \sqrt{100 - 4(\frac{50}{3})} \approx 5.77 \text{ cm}$

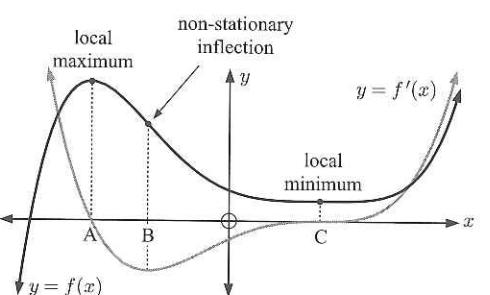
7 At $x = B$, $f''(x) = 0$ but $f'(x) \neq 0$

∴ $f(x)$ has a non-stationary inflection point at $x = B$.

$f'(x)$ is above the x -axis for $x \leq A$ and $x \geq C$, and below the x -axis for $A \leq x \leq C$

∴ $f(x)$ is increasing for $x \leq A$, decreasing for $A \leq x \leq C$, then increasing for $x \geq C$

∴ $f(x)$ has a local maximum at $x = A$ and a local minimum at $x = C$.



Chapter 19

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXERCISE 19A

- 1 a $f(x) = e^{4x}$
 $\therefore f'(x) = 4e^{4x}$
- b $f(x) = e^x + 3$
 $\therefore f'(x) = e^x + 0$
 $= e^x$
 $\therefore f'(x) = -2e^{-2x}$
- c $f(x) = \exp(-2x)$
 $= e^{-2x}$
- d $f(x) = e^{\frac{x}{2}}$
 $\therefore f'(x) = \frac{1}{2}e^{\frac{x}{2}}$
- e $f(x) = 2e^{-\frac{x}{2}}$
 $\therefore f'(x) = 2e^{-\frac{x}{2}} (-\frac{1}{2})$
 $= -e^{-\frac{x}{2}}$
 $\therefore f'(x) = 0 - 2e^{-x}(-1)$
 $= 2e^{-x}$
- g $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$
 $\therefore f'(x) = 4e^{\frac{x}{2}} (\frac{1}{2}) - 3e^{-x}(-1)$
 $= 2e^{\frac{x}{2}} + 3e^{-x}$
- h $f(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + e^{-x})$
 $\therefore f'(x) = \frac{1}{2}(e^x + e^{-x}(-1))$
 $= \frac{1}{2}(e^x - e^{-x})$
- i $f(x) = e^{-x^2}$
 $\therefore f'(x) = e^{-x^2}(-2x)$
 $= -2xe^{-x^2}$
- j $f(x) = e^{\frac{1}{x}}$
 $\therefore f'(x) = e^{\frac{1}{x}} \left(-\frac{1}{x^2}\right)$
 $= -\frac{e^{\frac{1}{x}}}{x^2}$
- k $f(x) = 10(1 + e^{2x})$
 $= 10 + 10e^{2x}$
- l $f(x) = 20(1 - e^{-2x})$
 $= 20 - 20e^{-2x}$
- m $f(x) = e^{2x+1}$
 $\therefore f'(x) = e^{2x+1}(2)$
 $= 2e^{2x+1}$
- n $f(x) = e^{\frac{x}{4}}$
 $\therefore f'(x) = e^{\frac{x}{4}}(\frac{1}{4})$
 $= \frac{1}{4}e^{\frac{x}{4}}$
- o $f(x) = e^{1-2x^2}$
 $\therefore f'(x) = e^{1-2x^2}(-4x)$
 $= -4xe^{1-2x^2}$
- p $f(x) = e^{-0.02x}$
 $\therefore f'(x) = e^{-0.02x} \times (-0.02)$
 $= -0.02e^{-0.02x}$
- 2 a $f(x) = xe^x$
 $\therefore f'(x) = 1e^x + xe^x \quad \{\text{product rule}\}$
 $= e^x + xe^x$
- b $f(x) = x^3 e^{-x}$
 $\therefore f'(x) = 3x^2 e^{-x} + x^3 (-e^{-x})$
 $\quad \{\text{product rule}\}$
 $= 3x^2 e^{-x} - x^3 e^{-x}$
- c $f(x) = \frac{e^x}{x}$
 $\therefore f'(x) = \frac{e^x x - e^x (1)}{x^2} \quad \{\text{quotient rule}\}$
 $= \frac{xe^x - e^x}{x^2}$
- d $f(x) = \frac{x}{e^x}$
 $\therefore f'(x) = \frac{1e^x - xe^x}{(e^x)^2} \quad \{\text{quotient rule}\}$
 $= \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x}$
- e $f(x) = x^2 e^{3x}$
 $\therefore f'(x) = 2xe^{3x} + 3x^2 e^{3x} \quad \{\text{product rule}\}$
- f $f(x) = \frac{e^x}{\sqrt{x}}$
 $\therefore f'(x) = \frac{e^x \sqrt{x} - \frac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2} \quad \{\text{quotient rule}\}$
 $= \frac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g $f(x) = \sqrt{x}e^{-x}$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^{-x} - x^{\frac{1}{2}}e^{-x}$$

$$= \frac{1}{2\sqrt{x}}e^{-x} - \sqrt{x}e^{-x}$$

{product rule}

h $f(x) = \frac{e^x + 2}{e^{-x} + 1}$

$$\therefore f'(x) = \frac{e^x(e^{-x} + 1) - (e^x + 2)(-e^{-x})}{(e^{-x} + 1)^2}$$

$$= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x} + 1)^2}$$

$$= \frac{e^x + 2 + 2e^{-x}}{(e^{-x} + 1)^2}$$

{quotient rule}

3 a $f(x) = (e^x + 2)^4$

$$= u^4 \text{ where } u = e^x + 2$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \text{chain rule}$$

$$= 4u^3(e^x)$$

$$\therefore f'(x) = 4(e^x + 2)^3(e^x)$$

$$= 4e^x(e^x + 2)^3$$

c $f(x) = \sqrt{e^{2x} + 10}$

$$= u^{\frac{1}{2}} \text{ where } u = e^{2x} + 10$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \text{chain rule}$$

$$= \frac{1}{2}u^{-\frac{1}{2}}(2e^{2x})$$

$$\therefore f'(x) = \frac{e^{2x}}{\sqrt{e^{2x} + 10}}$$

e $f(x) = \frac{1}{\sqrt{1 - e^{-x}}}$

$$= u^{-\frac{1}{2}} \text{ where } u = 1 - e^{-x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \text{chain rule}$$

$$= -\frac{1}{2}u^{-\frac{3}{2}}(e^{-x})$$

$$= \frac{-e^{-x}}{2u^{\frac{3}{2}}}$$

$$\therefore f'(x) = \frac{-e^{-x}}{2(1 - e^{-x})^{\frac{3}{2}}}$$

4 $y = Ae^{kx}$

a $\frac{dy}{dx} = Ae^{kx}(k)$

$$= k(Ae^{kx})$$

$$= ky$$

b $f(x) = \frac{1}{1 - e^{-x}}$

$$= u^{-1} \text{ where } u = 1 - e^{-x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \text{chain rule}$$

$$= -u^{-2}(e^{-x})$$

$$\therefore f'(x) = -\frac{e^{-x}}{(1 - e^{-x})^2}$$

d $f(x) = \frac{1}{(1 - e^{3x})^2}$

$$= u^{-2} \text{ where } u = 1 - e^{3x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \text{chain rule}$$

$$= -2u^{-3}(-3e^{3x}) = \frac{6e^{3x}}{u^3}$$

$$\therefore f'(x) = \frac{6e^{3x}}{(1 - e^{3x})^3}$$

f $f(x) = x\sqrt{1 - 2e^{-x}}$

$$= xu^{\frac{1}{2}} \text{ where } u = 1 - 2e^{-x}$$

$$\therefore f'(x) = 1u^{\frac{1}{2}} + x \times \frac{1}{2}u^{-\frac{1}{2}} \frac{du}{dx}$$

{product rule and chain rule}

$$= 1\sqrt{u} + x \frac{1}{2}u^{-\frac{1}{2}}2e^{-x}$$

$$= \frac{\sqrt{1 - 2e^{-x}}}{1} + \frac{xe^{-x}}{\sqrt{1 - 2e^{-x}}}$$

$$\therefore f'(x) = \frac{1 - 2e^{-x} + xe^{-x}}{\sqrt{1 - 2e^{-x}}}$$

b $\frac{d^2y}{dx^2} = \frac{d}{dx} kAe^{kx} \quad \text{from a}$

$$= k^2Ae^{kx}$$

$$= k^2y$$

5 $y = 2e^{3x} + 5e^{4x} \quad \therefore \frac{dy}{dx} = 6e^{3x} + 20e^{4x} \text{ and } \frac{d^2y}{dx^2} = 18e^{3x} + 80e^{4x}$

Now $\frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 12y = (18e^{3x} + 80e^{4x}) - 7(6e^{3x} + 20e^{4x}) + 12(2e^{3x} + 5e^{4x})$
 $= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x}$
 $= e^{3x}[18 - 42 + 24] + e^{4x}[80 - 140 + 60]$
 $= e^{3x}(0) + e^{4x}(0)$
 $= 0$
 $\therefore \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 12y = 0$

6 $f(x) = e^{kx} + x \quad \therefore f'(x) = ke^{kx} + 1$

Now $f'(0) = -8$, so $ke^0 + 1 = -8$
 $\therefore k \times 1 = -9$
 $\therefore k = -9$

7 a $y = xe^{-x}$

$$\begin{aligned} \frac{dy}{dx} &= 1e^{-x} - xe^{-x} && \text{product rule} \\ &= e^{-x}(1 - x) \\ &= \frac{1 - x}{e^x} \end{aligned}$$

which has sign diagram:

When $x = 1$, $y = 1e^{-1} = \frac{1}{e}$, so we have a local maximum at $(1, \frac{1}{e})$.

b $y = x^2e^x$

$$\begin{aligned} \frac{dy}{dx} &= 2xe^x + x^2e^x && \text{product rule} \\ &= xe^x(2 + x) \end{aligned}$$

which has sign diagram:

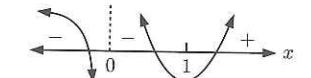
When $x = -2$, $y = 4e^{-2}$, and when $x = 0$, $y = 0$.

So, we have a local maximum at $(-2, \frac{4}{e^2})$, and a local minimum at $(0, 0)$.

c $y = \frac{e^x}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x x - e^x (1)}{x^2} && \text{quotient rule} \\ &= \frac{e^x(x - 1)}{x^2} \end{aligned}$$

which has sign diagram:



When $x = 1$, $y = \frac{e^1}{1} = e$, so we have a local minimum at $(1, e)$.

d $y = e^{-x}(x + 2)$

$$\begin{aligned} \frac{dy}{dx} &= -e^{-x}(x + 2) + e^{-x} && \text{product rule} \\ &= e^{-x}(-x - 2 + 1) \\ &= e^{-x}(-x - 1) \end{aligned}$$

which has sign diagram:

When $x = -1$, $y = e(-1 + 2) = e$, so we have a local maximum at $(-1, e)$.

The y -intercepts occur when $x = 0$

$$\text{Now } f(0) = e^0 - 3 = -2 \text{ and } g(0) = 3 - 5 = -2$$

\therefore both $f(x)$ and $g(x)$ have y -intercept -2 .

b As $x \rightarrow \infty$, $f(x) \rightarrow \infty$

$x \rightarrow -\infty$, $f(x) \rightarrow -3$ (above)

As $x \rightarrow \infty$, $g(x) \rightarrow 3$ (below)

$x \rightarrow -\infty$, $g(x) \rightarrow -\infty$

c $f(x)$ and $g(x)$ meet when

$$e^x - 3 = 3 - 5e^{-x}$$

$$\therefore e^{2x} - 3e^x = 3e^x - 5 \quad \{\times e^x\}$$

$$\therefore e^{2x} - 6e^x + 5 = 0$$

$$\therefore (e^x - 5)(e^x - 1) = 0$$

$$\therefore e^x = 5 \text{ or } 1$$

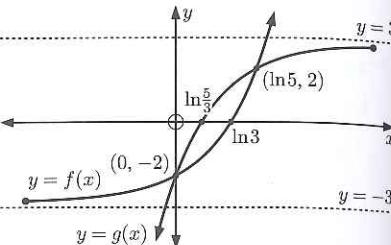
$$\therefore x = \ln 5 \text{ or } 0$$

$$\text{Now } f(\ln 5) = e^{\ln 5} - 3 = 5 - 3 = 2$$

and $f(0) = -2$

$\therefore f(x)$ and $g(x)$ meet at $(0, -2)$ and $(\ln 5, 2)$.

d



7 a Consider $y = e^x - 3e^{-x}$

It cuts the x -axis at P when $y = 0$

$$\therefore e^x - 3e^{-x} = 0$$

$$\therefore e^{2x} - 3 = 0 \quad \{\times e^x\}$$

$$\therefore e^{2x} = 3$$

$$\therefore 2x = \ln 3$$

$$\therefore x = \frac{1}{2} \ln 3$$

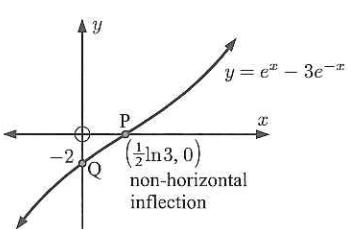
$$\frac{dy}{dx} = e^x + 3e^{-x}$$

$$= e^x + \frac{3}{e^x}$$

Since $e^x > 0$ for all x ,

$$\frac{dy}{dx} > 0 \text{ for all } x$$

\therefore the function is increasing for all x



$$\frac{dy}{dx} = e^x + 3e^{-x}$$

$$\therefore \frac{d^2y}{dx^2} = e^x - 3e^{-x}$$

$$= y$$

Above the x -axis $y > 0 \therefore \frac{d^2y}{dx^2} > 0$

\therefore the function is concave up

Below the x -axis $y < 0 \therefore \frac{d^2y}{dx^2} < 0$

\therefore the function is concave down

\therefore a non-horizontal inflection occurs when $y = 0$

EXERCISE 19C

1 a $y = \ln(7x)$ or $y = \ln(7x)$

$$\therefore y = \ln 7 + \ln x$$

$$\therefore \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{7}{7x} \leftarrow f'(x)$$

$$= \frac{1}{x}$$

b $y = \ln(2x + 1)$

$$\therefore \frac{dy}{dx} = \frac{2}{2x + 1} \leftarrow f'(x)$$

c $y = \ln(x - x^2)$

$$\therefore \frac{dy}{dx} = \frac{1 - 2x}{x - x^2} \leftarrow f'(x)$$

d $y = 3 - 2 \ln x$

$$\therefore \frac{dy}{dx} = 0 - 2 \left(\frac{1}{x} \right)$$

$$= -\frac{2}{x}$$

f $y = \frac{\ln x}{2x}$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{1}{x} \right) 2x - \ln x \times 2}{(2x)^2}$$

$$= \frac{2 - 2 \ln x}{4x^2}$$

$$= \frac{1 - \ln x}{2x^2}$$

i $y = \sqrt{\ln x} = (\ln x)^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \left(\frac{1}{x} \right)$$

$$= \frac{1}{2x\sqrt{\ln x}}$$

k $y = \sqrt{x} \ln(2x)$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \ln(2x) + \sqrt{x} \left(\frac{1}{x} \right)$$

$$= \frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

m $y = 3 - 4 \ln(1 - x)$

$$\therefore \frac{dy}{dx} = -\frac{4}{1-x} \times -1$$

$$= \frac{4}{1-x}$$

2 a $y = x \ln 5$

$$\therefore \frac{dy}{dx} = \ln 5$$

c $y = \ln(x^4 + x)$

$$\therefore \frac{dy}{dx} = \frac{4x^3 + 1}{x^4 + x}$$

b $y = \ln(x^3) = 3 \ln x$

$$\therefore \frac{dy}{dx} = 3 \left(\frac{1}{x} \right) = \frac{3}{x}$$

d $y = \ln(10 - 5x)$

$$\therefore \frac{dy}{dx} = \frac{-5}{10 - 5x} = \frac{1}{x - 2}$$

e $y = [\ln(2x+1)]^3$

$$\therefore \frac{dy}{dx} = 3[\ln(2x+1)]^2 \times \frac{2}{2x+1}$$

$$= \frac{6[\ln(2x+1)]^2}{2x+1}$$

f $y = \frac{\ln(4x)}{x}$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2}$$

$$= \frac{1 - \ln(4x)}{x^2}$$

g $y = \ln\left(\frac{1}{x}\right)$

$$= -\ln x$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x}$$

h $y = \ln(\ln x)$

$$\therefore \frac{dy}{dx} = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \ln x}$$

i $y = \frac{1}{\ln x} = (\ln x)^{-1}$

$$\therefore \frac{dy}{dx} = -1(\ln x)^{-2} \times \frac{1}{x}$$

$$= \frac{-1}{x(\ln x)^2}$$

3 a $y = \ln\sqrt{1-2x}$

$$= \ln(1-2x)^{\frac{1}{2}}$$

$$= \frac{1}{2}\ln(1-2x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \times \frac{-2}{1-2x}$$

$$= \frac{-1}{1-2x}$$

$$= \frac{1}{2x-1}$$

b $y = \ln\left(\frac{1}{2x+3}\right)$

$$= -\ln(2x+3)$$

$$\therefore \frac{dy}{dx} = -\frac{2}{2x+3}$$

c $y = \ln(e^x \sqrt{x})$

$$= \ln e^x + \ln x^{\frac{1}{2}}$$

$$= \ln e^x + \frac{1}{2}\ln x$$

$$= x + \frac{1}{2}\ln x$$

$$\therefore \frac{dy}{dx} = 1 + \frac{1}{2}\left(\frac{1}{x}\right)$$

$$= 1 + \frac{1}{2x}$$

d $y = \ln(x\sqrt{2-x})$

$$= \ln x + \ln(2-x)^{\frac{1}{2}}$$

$$= \ln x + \frac{1}{2}\ln(2-x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{-1}{2-x}\right)$$

$$= \frac{1}{x} - \frac{1}{2(2-x)}$$

g $f(x) = \ln((3x-4)^3)$

$$= 3\ln(3x-4)$$

$$\therefore f'(x) = 3 \times \frac{3}{3x-4}$$

$$= \frac{9}{3x-4}$$

e $y = \ln\left(\frac{x+3}{x-1}\right)$

$$= \ln(x+3) - \ln(x-1)$$

$$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{1}{x-1}$$

f $y = \ln\left(\frac{x^2+2x}{3-x}\right)$

$$= \ln x^2 - \ln(3-x)$$

$$= 2\ln x - \ln(3-x)$$

$$\therefore \frac{dy}{dx} = \frac{2}{x} - \frac{-1}{3-x}$$

$$= \frac{2}{x} + \frac{1}{3-x}$$

4 a $y = 2^x$

$$= (e^{\ln 2})^x$$

$$= e^{x \ln 2}$$

$$\therefore \frac{dy}{dx} = e^{x \ln 2} \times \ln 2$$

$$= 2^x \ln 2$$

b $y = a^x$

$$= (e^{\ln a})^x$$

$$= e^{x \ln a}$$

$$\therefore \frac{dy}{dx} = e^{x \ln a} \times \ln a$$

$$= a^x \ln a$$

5 $f(x) = \ln(2x-1) - 3$

a $f(x) = 0$ when $\ln(2x-1) = 3$

$$\therefore 2x-1 = e^3$$

$$\therefore 2x = e^3 + 1$$

$$\therefore x = \frac{e^3 + 1}{2} \approx 10.5 \quad \therefore \text{the } x\text{-intercept is } \frac{e^3 + 1}{2}$$

b $f(0)$ cannot be found as $\ln(-1)$ is not defined. \therefore there is no y -intercept.

c $f'(x) = \frac{2}{2x-1} \quad \therefore f'(1) = \frac{2}{2-1} = 2 \quad \therefore \text{gradient of tangent} = 2$

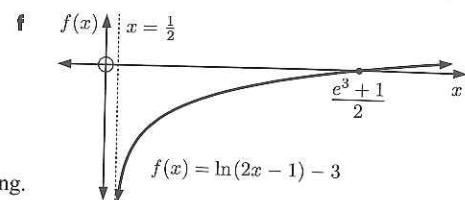
d $\ln(2x-1)$ has meaning provided $2x-1 > 0 \quad \therefore 2x > 1$ and so $x > \frac{1}{2}$
 $\therefore f(x)$ has meaning provided $x > \frac{1}{2}$

e $f'(x) = 2(2x-1)^{-1}$

$$\therefore f''(x) = -2(2x-1)^{-2}(2)$$

$$= \frac{-4}{(2x-1)^2}, \quad x > \frac{1}{2}$$

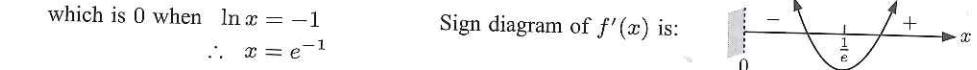
\therefore provided $x > \frac{1}{2}$, $f''(x) < 0$
 $\therefore f(x)$ is concave down when $f(x)$ has meaning.



6 a $f(x)$ is defined when $\ln x$ is defined $\therefore f(x)$ is defined for $x > 0$

b $f'(x) = \ln x + \frac{x}{x}$ {product rule}
 $= \ln x + 1$

which is 0 when $\ln x = -1$



So, there is a local minimum at $(\frac{1}{e}, \frac{1}{e} \ln \frac{1}{e})$
 \therefore the minimum value of $f(x)$ is $\frac{1}{e} \ln e^{-1} = -\frac{1}{e}$

7 Consider $f(x) = \frac{\ln x}{x}$

$$\therefore f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$\therefore f'(x) = 0$ when $1 - \ln x = 0$

$$\therefore \ln x = 1 \quad \therefore x = e$$

Now $f(e) = \frac{\ln e}{e} = \frac{1}{e}$
 \therefore there is a local maximum at $(e, \frac{1}{e})$

$\therefore f(x) \leq \frac{1}{e}$ for all x , and so $\frac{\ln x}{x} \leq \frac{1}{e}$ for all $x > 0$

8 a $f(x) = x - \ln x$

$$\therefore f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x} \quad \text{and the sign diagram of } f'(x) \text{ is:}$$

$\therefore f(x)$ has a local minimum at $(1, 1 - \ln 1)$ or $(1, 1)$. This is the only turning point.

b $f(x) \geq 1$ for all $x > 0$

$$\therefore x - \ln x \geq 1$$

$\therefore \ln x \leq x - 1$ for all $x > 0$

