

## EXERCISE 19D

1  $f(x) = e^{-x}$

$\therefore f(1) = e^{-1}$

$\therefore$  the point of contact is  $(1, \frac{1}{e})$ .

Now  $f'(x) = -e^{-x}$

$\therefore f'(1) = -e^{-1} = -\frac{1}{e}$

So, the gradient of the tangent is  $-\frac{1}{e}$

2  $y = \ln(2-x)$

so when  $x = -1$ ,  $y = \ln 3$

$\therefore$  the point of contact is  $(-1, \ln 3)$ .

Now  $\frac{dy}{dx} = \frac{-1}{2-x}$

$\therefore$  when  $x = -1$ ,  $\frac{dy}{dx} = -\frac{1}{2+1} = -\frac{1}{3}$

So, the gradient of the tangent is  $-\frac{1}{3}$ .

3  $y = x^2 e^x$  so when  $x = 1$ ,  $y = e$

$\therefore$  the point of contact is  $(1, e)$ .

Now  $\frac{dy}{dx} = 2xe^x + x^2 e^x$

$\therefore$  when  $x = 1$ ,  $\frac{dy}{dx} = 2e + e = 3e$

$\therefore$  the tangent has equation  $\frac{y-e}{x-1} = 3e$

$\therefore y - e = 3ex - 3e$

$\therefore y - 3ex = -2e$

$\therefore 3ex - y = 2e$

4  $y = \ln \sqrt{x}$   $\therefore$  when  $y = -1$ ,  $-1 = \frac{1}{2} \ln x$

$= \ln x^{\frac{1}{2}}$   $\therefore \ln x = -2$

$= \frac{1}{2} \ln x$   $\therefore x = e^{-2}$

$\therefore x = \frac{1}{e^2}$   $\therefore$  the point of contact is  $(\frac{1}{e^2}, -1)$

Now  $\frac{dy}{dx} = \frac{1}{2} \frac{1}{x} = \frac{1}{2x}$ , so at the point of contact,  $\frac{dy}{dx} = \frac{1}{2e^{-2}} = \frac{e^2}{2}$

$\therefore$  the tangent has gradient  $\frac{e^2}{2}$  and the normal has gradient  $-\frac{2}{e^2}$

$\therefore$  the normal has equation  $\frac{y+1}{x - \frac{1}{e^2}} = -\frac{2}{e^2}$

$\therefore e^2(y+1) = -2\left(x - \frac{1}{e^2}\right)$

$\therefore e^2y + e^2 = -2x + \frac{2}{e^2}$

$\therefore 2x + e^2y = -e^2 + \frac{2}{e^2}$  or  $y = -\frac{2}{e^2}x + \frac{2}{e^4} - 1$

The tangent cuts the  $x$ -axis when

$y = 0$

$\therefore 3ex = 2e$

$\therefore x = \frac{2}{3}$

and the  $y$ -axis when

$x = 0$

$\therefore -y = 2e$

$\therefore y = -2e$

So, A is  $(\frac{2}{3}, 0)$  and B is  $(0, -2e)$ .

5  $y = e^x$  so when  $x = a$ ,  $y = e^a$   
 $\therefore$  the point of contact is  $(a, e^a)$ .

Now  $\frac{dy}{dx} = e^x$

$\therefore$  at the point  $(a, e^a)$ ,  $\frac{dy}{dx} = e^a$

$\therefore$  the tangent has equation  $\frac{y - e^a}{x - a} = e^a$   
 or  $y - e^a = e^a(x - a)$  .... (\*)

Since the tangent passes through the origin,  
 $(0, 0)$  must satisfy (\*).

$\therefore 0 - e^a = e^a(0 - a)$   
 $\therefore -e^a = -ae^a$

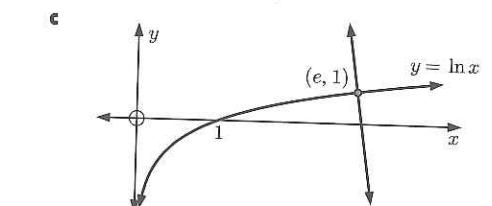
$\therefore e^a(a - 1) = 0$

$\therefore a = 1$  {as  $e^a > 0$ }  
 So the equation of the tangent is  
 $y - e = ex - e$  or  $y = ex$ .

6 a  $f(x) = \ln x$  is defined for all  $x > 0$ .

b  $f'(x) = \frac{1}{x}$  which is  $> 0$  for all  $x > 0$

$\therefore f(x)$  is increasing on  $x > 0$ ; its gradient is always positive.  
 $f''(x) = -x^{-2} = \frac{-1}{x^2}$  which is  $< 0$  for all  $x > 0$   $\therefore f(x)$  is concave down on  $x > 0$ .



At  $y = 1$ ,  $1 = \ln x$   
 $\therefore x = e^1 = e$

$\therefore$  the point of contact is  $(e, 1)$

Now  $\frac{dy}{dx} = \frac{1}{x}$

$\therefore$  at  $(e, 1)$ ,  $\frac{dy}{dx} = \frac{1}{e}$

$\therefore$  the gradient of the tangent is  $\frac{1}{e}$ , and the gradient of the normal is  $-e$

$\therefore$  the equation of the normal is  $\frac{y-1}{x-e} = -e$   $\therefore y-1 = -e(x-e)$

$\therefore y-1 = -ex + e^2$   
 $\therefore ex + y = 1 + e^2$

7  $y = 3e^{-x}$  and  $y = 2 + e^x$  meet when  $3e^{-x} = 2 + e^x$

$\therefore 3 = 2e^x + e^{2x}$  { $\times e^x$ }  
 $\therefore e^{2x} + 2e^x - 3 = 0$

$\therefore (e^x + 3)(e^x - 1) = 0$

$\therefore e^x = -3$  or 1

$\therefore e^x = 1$  and so  $x = 0$  {as  $e^x > 0$ }

Now when  $x = 0$ ,  $y = 3e^0 = 3$ , so the graphs meet at  $(0, 3)$ .

For  $y = 2 + e^x$ ,  $\frac{dy}{dx} = e^x$ ,

For  $y = 3e^{-x}$ ,  $\frac{dy}{dx} = -3e^{-x}$ ,

so at the point  $(0, 3)$ ,  $\frac{dy}{dx} = e^0 = 1$

so at the point  $(0, 3)$ ,  $\frac{dy}{dx} = -3$

$\therefore$  the gradient of the tangent at this point is 1

$\therefore$  the gradient of the tangent at this point is  $-3$

$\therefore$  the tangent has direction vector  $(1, 1)$

$\therefore$  the tangent has direction vector  $(1, -3)$

If  $\theta$  is the acute angle between the tangents, then  $\cos \theta = \frac{|1(1) + 1(-3)|}{\sqrt{1^2 + 1^2} \sqrt{1^2 + (-3)^2}} = \frac{|-2|}{\sqrt{2} \sqrt{10}} = \frac{2}{\sqrt{20}}$   
 $\therefore \theta \approx 63.43^\circ$

8 a  $W = 20e^{-kt}$  so when  $t = 50$  hours,  $W = 10$  g

$\therefore 20e^{-50k} = 10$

$\therefore e^{-50k} = \frac{1}{2}$

$\therefore -50k = \ln \frac{1}{2} = -\ln 2 \quad \therefore k = \frac{1}{50} \ln 2 \approx 0.0139$

**b** i When  $t = 0$ ,  
 $W = 20e^0 = 20$  g

ii When  $t = 24$ ,  
 $W = 20e^{-24k} = 20e^{-24 \frac{\ln 2}{50}} \approx 14.3$  g

iii When  $t = 1$  week  
 $= 7 \times 24$  hours  
 $= 168$  hours  
 $W = 20e^{-168 \frac{\ln 2}{50}} \approx 1.95$  g

c When  $W = 1$  g,  $20e^{-\frac{\ln 2}{50} \times t} = 1$

$$\therefore e^{-\frac{\ln 2}{50} \times t} = 0.05$$

$$\therefore -\frac{\ln 2}{50} \times t = \ln 0.05$$

$$\therefore t = \frac{-50 \ln 0.05}{\ln 2} \approx 216 \text{ hours or 9 days and 6 minutes}$$

d  $\frac{dW}{dt}$   
 $= 20e^{-kt}(-k)$   
 $= \left(-20 \frac{\ln 2}{50}\right) \times e^{-\frac{\ln 2}{50}t}$

e  $\frac{dW}{dt} = -k(20e^{-kt}) = -kW \quad \therefore \frac{dW}{dt} \propto W$

9  $T = 5 + 95e^{-kt} \text{ } ^\circ\text{C}$

a  $T = 20 \text{ } ^\circ\text{C}$  when  $t = 15$   
 $20 = 5 + 95e^{-15k}$   
 $\therefore 15 = 95e^{-15k}$   
 $\therefore e^{15k} = \frac{95}{15}$

$$\therefore 15k = \ln\left(\frac{19}{3}\right)$$

$$\therefore k = \frac{1}{15} \ln\left(\frac{19}{3}\right) \approx 0.123$$

d  $\frac{dT}{dt} = -95e^{-kt} \times k \approx -11.6902e^{-0.1231t}$

- i When  $t = 0$ ,  $\frac{dT}{dt} \approx -11.69$ , so the temperature is decreasing at  $11.7 \text{ } ^\circ\text{C min}^{-1}$ .
- ii When  $t = 10$ ,  $\frac{dT}{dt} \approx -11.6902e^{-1.231} \approx -3.415$ , so the temperature is decreasing at  $3.42 \text{ } ^\circ\text{C min}^{-1}$ .
- iii When  $t = 20$ ,  $\frac{dT}{dt} \approx -11.6902e^{-2.461} \approx -0.998$ , so the temperature is decreasing at  $0.998 \text{ } ^\circ\text{C min}^{-1}$ .

10  $H(t) = 20 \ln(3t+2) + 30 \text{ cm}, t \geq 0$

a The shrubs were planted when  $t = 0$ .  $H(0) = 20 \ln(2) + 30 \approx 43.9 \text{ cm}$

b When  $H = 1 \text{ m} = 100 \text{ cm}$ ,

$$20 \ln(3t+2) + 30 = 100$$

$$\therefore 20 \ln(3t+2) = 70$$

$$\therefore \ln(3t+2) = 3.5$$

$$\therefore 3t+2 = e^{3.5}$$

$$\therefore 3t = e^{3.5} - 2$$

$$\therefore t = \frac{e^{3.5} - 2}{3} \text{ years}$$

$$\therefore t \approx 10.4 \text{ years}$$

c  $\frac{dH}{dt} = 20 \times \frac{3}{(3t+2)} = \frac{60}{3t+2} \text{ cm year}^{-1}$

i When  $t = 3$ ,  $\frac{dH}{dt} = \frac{60}{11} \approx 5.4545$   
 $\therefore$  it is growing at  $5.45 \text{ cm year}^{-1}$

ii When  $t = 10$ ,  $\frac{dH}{dt} = \frac{60}{32} = 1.875$   
 $\therefore$  it is growing at  $1.88 \text{ cm year}^{-1}$

11 a  $A = s(1 - e^{-kt}), t \geq 0$   
 When  $t = 0$ ,  $A = s(1 - e^0) = s(1 - 1) = 0$

b When  $t = 3$ ,  $A = 5$  and  $s = 10$   
 $\therefore 5 = 10(1 - e^{-3k})$   
 $\therefore 0.5 = 1 - e^{-3k}$   
 $\therefore e^{-3k} = 0.5$   
 $\therefore e^{3k} = 2$   
 $\therefore 3k = \ln 2$   
 $\therefore k = \frac{1}{3} \ln 2 \approx 0.231$

c  $\frac{dA}{dt} = ske^{-kt}$

$\therefore$  when  $t = 5$  and  $s = 10$ ,

$$\frac{dA}{dt} = 10\left(\frac{1}{3} \ln 2\right)\left(e^{-\frac{5}{3} \ln 2}\right) \approx 0.728 \text{ litres per hour}$$

d  $\frac{dA}{dt} = ske^{-kt} = k(se^{-kt}) = -k(-se^{-kt}) = -k(A-s)$   
 $\therefore \frac{dA}{dt} \propto (A-s)$

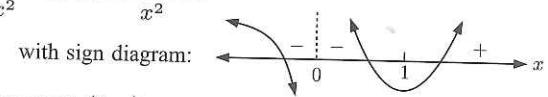
12 Consider  $f(x) = \frac{e^x}{x}$ .

a  $e^x \neq 0$  for all  $x$ , so  $f(x) \neq 0$  and there is no  $x$ -intercept.

$f(0) = \frac{e^0}{0}$  is undefined, so there is also no  $y$ -intercept.

b As  $x \rightarrow +\infty$ ,  $f(x) \rightarrow \infty$ , and as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0$  (below)  
 (As  $x \rightarrow 0$  (above),  $y \rightarrow +\infty$ , and as  $x \rightarrow 0$  (below),  $y \rightarrow -\infty$ )  
 $\therefore x = 0$  is a vertical asymptote.

c Using the quotient rule,  $f'(x) = \frac{e^x x - e^x(1)}{x^2} = \frac{e^x(x-1)}{x^2}$   
 with sign diagram:  
 $f(1) = \frac{e^1}{1} = e$ , so there is a local minimum at  $(1, e)$ .



d Now  $f'(x) = \frac{e^x(x-1)}{x^2}$   
 $\therefore f'(-1) = \frac{e^{-1}(-1-1)}{(-1)^2} = -\frac{2}{e}$   
 $\therefore$  the gradient of the tangent is  $-\frac{2}{e}$

When  $x = -1$ ,  $y = \frac{e^{-1}}{-1} = -\frac{1}{e}$   
 $\therefore$  the equation of tangent is  $\frac{y - \left(-\frac{1}{e}\right)}{x - (-1)} = -\frac{2}{e} \quad \therefore \frac{y + \frac{1}{e}}{x + 1} = -\frac{2}{e}$

$$\therefore e\left(y + \frac{1}{e}\right) = -2(x+1)$$

$$ey + 1 = -2x - 2$$

$$\therefore ey = -2x - 3$$

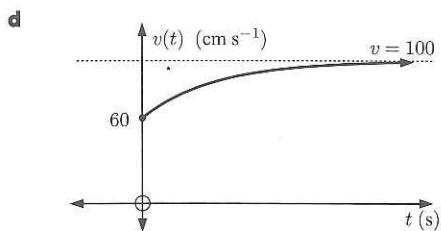
13 a  $s(t) = 100t + 200e^{-\frac{t}{5}} \text{ cm}, t \geq 0$

b When  $t = 0$ ,  $s(0) = 200 \text{ cm}$  (right of the origin)  
 $v(0) = 60 \text{ cm s}^{-1}$

$a(t) = 8e^{-\frac{t}{5}} \text{ cm s}^{-2}$

c As  $t \rightarrow \infty$ ,  $e^{-\frac{t}{5}} \rightarrow 0$ ,  
 $\therefore v(t) \rightarrow 100 \text{ cm s}^{-1}$  (below)

Teacher!



**14 a**  $A(t) = t \ln t + 1, 0 < t \leq 5$   
 $\therefore A'(t) = \ln t + t \times \frac{1}{t} + 0 \quad \{\text{product rule}\}$   
 $= \ln t + 1$

$\therefore A'(t) = 0 \text{ when } \ln t = -1$   
 $\therefore t = e^{-1}$

and the sign diagram of  $A'(t)$  is:

$\therefore A(t)$  is a minimum when  $t = \frac{1}{e} \approx 0.3679$  years

$\therefore$  the child's memorising ability is a minimum at 4.41 months old.

**15 a**  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$f'(x)$  has sign diagram:

$\therefore f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (-x)$

Now  $f(0) = \frac{1}{\sqrt{2\pi}}$

$= \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$   
 $\text{so there is a local maximum at } \left(0, \frac{1}{\sqrt{2\pi}}\right).$

$\therefore f'(x) = 0 \text{ when } x = 0$

The function is increasing for  $x \leq 0$  and decreasing for  $x \geq 0$

**b**  $f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} \left(-xe^{-\frac{1}{2}x^2}\right)$

$\therefore f''(x) = \frac{1}{\sqrt{2\pi}} \left((-1)e^{-\frac{1}{2}x^2} + (-x)e^{-\frac{1}{2}x^2}(-x)\right) \quad \{\text{product rule}\}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (x^2 - 1)$

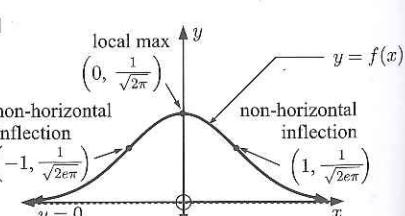
$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (x+1)(x-1) \text{ which has sign diagram:}$

Now  $f(1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sqrt{2e\pi}}$  and  $f(-1) = \frac{1}{\sqrt{2e\pi}}$

$\therefore$  there are points of inflection at  $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$  and  $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$ .

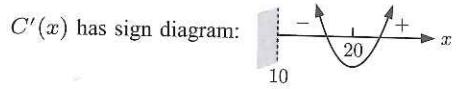
**c** As  $x \rightarrow \infty, e^{-\frac{1}{2}x^2} \rightarrow 0$  (above),  
 $\therefore f(x) \rightarrow 0$  (above)

As  $x \rightarrow -\infty, e^{-\frac{1}{2}x^2} \rightarrow 0$  (above),  
 $\therefore f(x) \rightarrow 0$  (above)



**16**  $C(x) = 4 \ln x + \left(\frac{30-x}{10}\right)^2, x \geq 10$

$$\begin{aligned} \therefore C'(x) &= \frac{4}{x} + 2 \left(\frac{30-x}{10}\right) \left(-\frac{1}{10}\right) \\ &= \frac{4}{x} - \frac{30-x}{50} \\ &= \frac{200-x(30-x)}{50x} \\ &= \frac{200-30x+x^2}{50x} \\ &= \frac{(x-10)(x-20)}{50x} \end{aligned}$$

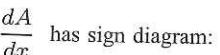


$\therefore$  the minimum cost occurs when  $x = 20$  or when 20 kettles per day are produced.

**17** Let coordinates of D be  $(x, 0)$  where  $x > 0$ .

$\therefore$  the coordinates of C are  $(x, e^{-x^2})$ .

$$\begin{aligned} \text{area ABCD} &= 2xe^{-x^2} \\ \therefore \frac{dA}{dx} &= 2e^{-x^2} + 2xe^{-x^2}(-2x) \quad \{\text{product rule}\} \\ &= 2e^{-x^2}(1-2x^2) \\ &= 2e^{-x^2}(1+\sqrt{2}x)(1-\sqrt{2}x) \end{aligned}$$



$\therefore$  the area is a maximum when  $x = \frac{1}{\sqrt{2}}$  and so C is  $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ .

**18**  $P(x) = R(x) - C(x)$

$$\begin{aligned} \therefore P(x) &= \left[1000 \ln \left(1 + \frac{x}{400}\right) + 600\right] - [x(1.5) + 300] \\ &= 1000 \ln(1 + 0.0025x) - 1.5x + 300 \end{aligned}$$

$$\therefore P'(x) = 1000 \left(\frac{0.0025}{1 + 0.0025x}\right) - 1.5 = \frac{2.5}{1 + 0.0025x} - 1.5$$

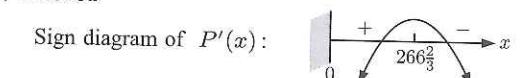
$$\therefore P'(x) = 0 \text{ when } \frac{2.5}{1 + 0.0025x} = \frac{3}{2}$$

$$\therefore 3 + 0.0075x = 5$$

$$\therefore 0.0075x = 2$$

$$\therefore x = \frac{2}{0.0075} \approx 266.7$$

Now  $P(266) \approx 410.83$  and  $P(267) \approx 410.83$   
 $\therefore$  to maximise the profit, 266 or 267 torches per day should be produced.



**19 a**  $y = ax^2, a > 0$  touches  $y = \ln x$  when  $ax^2 = \ln x$

If the curves touch when  $x = b$  then  $ab^2 = \ln b \dots (1)$

$$\text{Now for } y = ax^2, \frac{dy}{dx} = 2ax \quad \text{and for } y = \ln x, \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \text{when } x = b, \frac{dy}{dx} = 2ab \quad \therefore \text{when } x = b, \frac{dy}{dx} = \frac{1}{b}$$

Since the curves touch each other, they share a common tangent.

$$\therefore \frac{1}{b} = 2ab \dots (2)$$

**b** Now  $ab^2 = \frac{1}{2}$  {from (2)}

and  $ab^2 = \ln b$  {from (1)}

$$\therefore \ln b = \frac{1}{2}$$

$$\therefore b = e^{\frac{1}{2}} = \sqrt{e}$$

$$\text{When } x = b = \sqrt{e}, y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$$

$\therefore$  the point of contact is  $(\sqrt{e}, \frac{1}{2})$ .

$$\therefore a = \frac{1}{2b^2} \quad \{\text{from (2)}\}$$

$$\therefore a = \frac{1}{2(\sqrt{e})^2} = \frac{1}{2e}$$

d The tangent has gradient  $\frac{1}{b} = \frac{1}{\sqrt{e}}$  and passes through  $(\sqrt{e}, \frac{1}{2})$

$$\therefore \text{the tangent is } \frac{y - \frac{1}{2}}{x - \sqrt{e}} = \frac{1}{\sqrt{e}} \quad \therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}(x - \sqrt{e}) \\ \therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}x - 1 \\ \therefore y = e^{-\frac{1}{2}}x - \frac{1}{2}$$

20  $P(t) = \frac{50000}{1 + 1000e^{-0.5t}}, \quad 0 \leq t \leq 25$   
 $= 50000(1 + 1000e^{-0.5t})^{-1}$

$$\therefore P'(t) = -50000(1 + 1000e^{-0.5t})^{-2}(-500e^{-0.5t}) \\ = 2.5 \times 10^7 e^{-0.5t}(1 + 1000e^{-0.5t})^{-2}$$

The wasp population is growing the fastest when  $\frac{dP}{dt}$  is a maximum.

Using technology, the graph of  $P'(t)$  can be drawn and the maximum obtained.  
 The maximum occurs when  $t \approx 13.8$  weeks.

21 Consider  $y = Ate^{-bt}, \quad t \geq 0, \quad A > 0, \quad b > 0$

- a This function has a  $y$ -intercept when  $t = 0$ , so  $y = A(0)e^{-b \times 0} = 0 \quad \therefore$  the  $y$ -intercept is 0.  
 It has a  $t$ -intercept when  $y = 0$ , so  $Ate^{-bt} = 0$   
 $\therefore t = 0$ , since  $e^{-bt} > 0$  for all  $t$ , and  $A > 0$   
 $\therefore$  the only  $t$ -intercept is 0.

b  $y = Ate^{-bt}$

$$\therefore \frac{dy}{dt} = Ae^{-bt} + At(-b)e^{-bt} \quad \{\text{product rule}\} \\ = (1 - bt)Ae^{-bt}$$

The function has stationary points when  $\frac{dy}{dt} = 0$ .

Since  $Ae^{-bt} > 0$  for all  $t$ , this occurs when  $1 - bt = 0$

$$\therefore t = \frac{1}{b}$$

The sign diagram of  $\frac{dy}{dt}$  is:

$\therefore$  there is a local maximum at  $\left(\frac{1}{b}, A \frac{1}{b} e^{-b \times \frac{1}{b}}\right)$ , or  $\left(\frac{1}{b}, \frac{A}{be}\right)$ .

c  $\frac{dy}{dt} = (1 - bt)Ae^{-bt}$

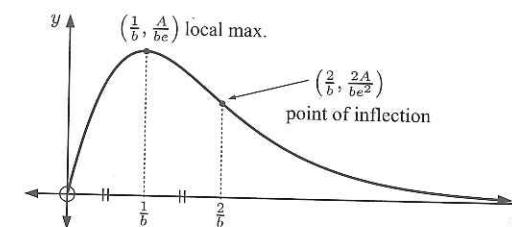
$$\therefore \frac{d^2y}{dt^2} = -bAe^{-bt} + (1 - bt)(-b)Ae^{-bt} \quad \{\text{product rule}\} \\ = Abe^{-bt}(tb - 2)$$

which is 0 when  $tb - 2 = 0 \quad \{Abe^{-bt} > 0\}$

$$\therefore t = \frac{2}{b}$$

At  $t = \frac{2}{b}$ ,  $y = A \frac{2}{b} e^{-b \times \frac{2}{b}} = \frac{2A}{be^2}$ .

$\therefore$  the point of inflection is  $\left(\frac{2}{b}, \frac{2A}{be^2}\right)$ . It is non-stationary, since  $\frac{dy}{dt} \neq 0$  at  $t = \frac{2}{b}$ .



- e  $E(t)$  has the form  $Ate^{-bt}$ , with  $A = 750$  and  $b = 1.5$ .  
 $\therefore$  the maximum occurs when  $t = \frac{1}{b} = \frac{1}{1.5} = \frac{2}{3}$  hour or 40 minutes

### REVIEW SET 19A

1 a  $y = e^{x^3+2} = e^u$  where  $u = x^3 + 2$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ = e^u \times 3x^2 \\ = 3x^2 e^u \\ = 3x^2 e^{x^3+2}$$

b  $y = \ln\left(\frac{x+3}{x^2}\right) = \ln(x+3) - \ln(x^2)$

$$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{2x}{x^2} \\ = \frac{1}{x+3} - \frac{2}{x}$$

2  $y = e^{-x^2}$  so when  $x = 1$ ,

$$y = e^{-1} = \frac{1}{e}$$

$\therefore$  the point of contact is  $\left(1, \frac{1}{e}\right)$

$\therefore$  the equation of the normal is  $\frac{y - \frac{1}{e}}{x - 1} = \frac{e}{2}$

Now  $\frac{dy}{dx} = -2xe^{-x^2}$

$$\therefore 2\left(y - \frac{1}{e}\right) = e(x - 1)$$

$\therefore$  when  $x = 1$ ,  $\frac{dy}{dx} = -2e^{-1}$

$$\therefore 2y - \frac{2}{e} = ex - e$$

$\therefore$  the gradient of the tangent is  $-\frac{2}{e}$

$$\therefore 2y = ex + \frac{2}{e} - e$$

and the gradient of the normal is  $\frac{e}{2}$

$$\therefore y = \frac{e}{2}x + \frac{1}{e} - \frac{e}{2}$$

3 a  $f(x)$  and  $g(x)$  intersect when  $e^{2x} = -e^x + 6$

$$\therefore (e^x)^2 + e^x - 6 = 0$$

$$\therefore (e^x + 3)(e^x - 2) = 0$$

$$\therefore e^x = -3 \text{ or } 2$$

But  $e^x > 0$  for all  $x$ , so  $e^x = 2$

$$\therefore x = \ln 2$$

$$f(\ln 2) = e^{2 \ln 2} = (e^{\ln 2})^2 = 4$$

$\therefore$  the point of intersection P is  $(\ln 2, 4)$ .

b  $f'(x) = 2e^{2x}$ , so the gradient of the tangent at P is  $f'(\ln 2) = 2 \times e^{2 \ln 2} = 8$

$$\therefore \text{the tangent has equation } \frac{y - 4}{x - \ln 2} = 8$$

$$\therefore y - 4 = 8x - 8 \ln 2$$

$$\therefore y = 8x + 4 - 8 \ln 2$$

**4** **a**  $f(x) = \frac{e^x}{x-1}$  has no  $x$ -intercepts since  $e^x$  is never 0.

Now  $f(0) = \frac{e^0}{-1} = -1$  so the  $y$ -intercept is  $-1$ .

**b**  $f(x)$  is defined for all  $x \neq 1$ .

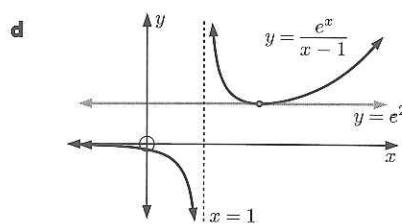
**c**  $f'(x) = \frac{e^x(x-1) - e^x(1)}{(x-1)^2}$  {quotient rule}

$$= \frac{e^x(x-2)}{(x-1)^2} \quad \text{and has sign diagram: } \begin{array}{c} \leftarrow - \end{array} \begin{array}{c} | \\ 1 \end{array} \begin{array}{c} - \end{array} \begin{array}{c} | \\ 2 \end{array} \begin{array}{c} + \end{array} \rightarrow x$$

$\therefore f(x)$  is decreasing for  $x < 1$  and  $1 < x \leq 2$ , and increasing for  $x \geq 2$ .

$$\begin{aligned} f''(x) &= \frac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1(1)]}{(x-1)^4} \quad \{\text{product and quotient rules}\} \\ &= \frac{[e^x(x-2+1)(x-1)^2] - 2e^x(x-2)(x-1)}{(x-1)^4} \\ &= \frac{e^x(x-1)(x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4} \\ &= \frac{e^x(x-1)[(x-1)^2 - 2(x-2)]}{(x-1)^4} \\ &= \frac{e^x(x-1)[x^2 - 2x + 1 - 2x + 4]}{(x-1)^4} \\ &= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3} \quad \text{where the quadratic term has } \Delta < 0 \end{aligned}$$

The sign diagram of  $f''(x)$  is:  $\begin{array}{c} \leftarrow - \end{array} \begin{array}{c} | \\ 1 \end{array} \begin{array}{c} + \end{array} \rightarrow x$   $\therefore f(x)$  is concave down for all  $x < 1$  and concave up for all  $x > 1$ .



**e** Now  $f(2) = \frac{e^2}{2-1} = e^2$

Using **c** we have a local minimum at  $(2, e^2)$ .  
 $\therefore$  the tangent at  $x = 2$  is horizontal and is  $y = e^2$ .

**5**  $y = \ln(x^2 + 3)$   $\therefore \frac{dy}{dx} = \frac{2x}{x^2 + 3}$

When  $x = 0$ ,  $\frac{dy}{dx} = 0$  so the gradient of the tangent at this point is 0.

But when  $x = 0$ ,  $y = \ln(0+3) = \ln 3$

$\therefore$  the tangent is  $y = \ln 3$  which does not cut the  $x$ -axis.

**6**  $f(x) = e^{4x} + px + q$

$\therefore f'(x) = 4e^{4x} + p$

At the point where  $x = 0$ , the tangent to  $f(x)$  has equation  $y = 5x - 7$ , so  $f'(0) = 5$

$$\therefore 4e^0 + p = 5 \\ \therefore p = 1$$

The tangent meets  $f(x)$  when  $x = 0$  and  $y = 5(0) - 7 = -7$ , so  $(0, -7)$  must lie on  $f(x)$  too.

$\therefore e^{4(0)} + p(0) + q = -7$

$\therefore 1 + q = -7$

$\therefore q = -8$

**7** **a**  $3e^x - 5 = -2e^{-x}$   
 $\therefore 3e^{2x} - 5e^x = -2 \quad \{\times e^x\}$   
 $\therefore 3e^{2x} - 5e^x + 2 = 0$   
 $\therefore (3e^x - 2)(e^x - 1) = 0$   
 $\therefore e^x = \frac{2}{3}$  or 1  
 $\therefore x = \ln \frac{2}{3}$  or 0

**b**  $2 \ln x - 3 \ln \left(\frac{1}{x}\right) = 10$   
 $\therefore 2 \ln x - 3 \ln(x^{-1}) = 10$   
 $\therefore 2 \ln x + 3 \ln x = 10$   
 $\therefore 5 \ln x = 10$   
 $\therefore \ln x = 2$   
 $\therefore x = e^2$

## REVIEW SET 19B

**1**  $H(t) = 60 + 40 \ln(2t+1)$  cm,  $t \geq 0$

**a** When first planted,  $t = 0 \therefore H(0) = 60 + 40 \ln(1) = 60 + 40(0) = 60$  cm.

**b** **i** When  $H(t) = 150$  cm,  
 $60 + 40 \ln(2t+1) = 150$   
 $40 \ln(2t+1) = 90$

$$\begin{aligned} \ln(2t+1) &= \frac{90}{40} = 2.25 \\ 2t+1 &= e^{2.25} \\ 2t &= e^{2.25} - 1 \\ t &= \frac{1}{2}(e^{2.25} - 1) \\ t &\approx 4.24 \text{ years} \end{aligned}$$

**ii** When  $H(t) = 300$  cm,  
 $60 + 40 \ln(2t+1) = 300$   
 $40 \ln(2t+1) = 240$

$$\begin{aligned} \ln(2t+1) &= \frac{240}{40} = 6 \\ 2t+1 &= e^6 \\ 2t &= e^6 - 1 \\ t &= \frac{1}{2}(e^6 - 1) \\ t &\approx 201 \text{ years} \end{aligned}$$

**c**  $H'(t) = 40 \left( \frac{2}{2t+1} \right) = \frac{80}{2t+1}$  cm per year

**i** When  $t = 2$ ,  $H'(2) = \frac{80}{5} = 16$  cm per year

**ii** When  $t = 20$ ,  $H'(20) = \frac{80}{41} \approx 1.95$  cm per year

**2**  $s(t) = 80e^{-\frac{t}{10}} - 40t$  metres,  $t \geq 0$

**a**  $v(t) = s'(t) = -8e^{-\frac{t}{10}} - 40$  m s<sup>-1</sup>

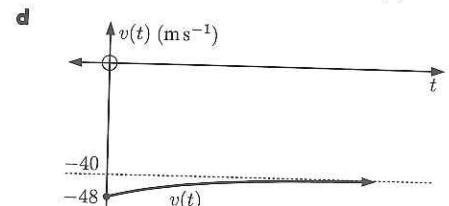
$a(t) = v'(t) = 0.8e^{-\frac{t}{10}}$  m s<sup>-2</sup>

**b** When  $t = 0$ ,  $s(0) = 80$  m

$v(0) = -48$  m s<sup>-1</sup>

$a(0) = 0.8$  m s<sup>-2</sup>

**c** As  $t \rightarrow \infty$ ,  $e^{-\frac{t}{10}} \rightarrow 0 \therefore v(t) \rightarrow -40$  m s<sup>-1</sup> (below)



**e** When  $v(t) = -44$  m s<sup>-1</sup>

$$-8e^{-\frac{t}{10}} - 40 = -44$$

$$-8e^{-\frac{t}{10}} = -4$$

$$e^{-\frac{t}{10}} = 0.5$$

$$-\frac{t}{10} = \ln 0.5$$

$$t = -10 \ln 0.5$$

$$\therefore t \approx 6.93 \text{ seconds}$$

**3**  $P(x) = R(x) - C(x)$

$$= \left[ 200 \ln \left( 1 + \frac{x}{100} \right) + 1000 \right] - [(x-100)^2 + 200]$$

$$= 200 \ln(1 + 0.01x) - (x-100)^2 + 800$$

$$\begin{aligned} \therefore \frac{dP}{dx} &= 200 \left( \frac{0.01}{1 + 0.01x} \right) - 2(x - 100)^1 \\ &= \frac{2}{1 + 0.01x} - \frac{2(x - 100)}{1} \\ &= \frac{2 - 2(x - 100)(1 + 0.01x)}{1 + 0.01x} \\ &= \frac{2 - 2(x + 0.01x^2 - 100 - x)}{1 + 0.01x} \\ &= \frac{2 - 0.02x^2 + 200}{1 + 0.01x} \\ &= \frac{202 - 0.02x^2}{1 + 0.01x} \end{aligned}$$

$\therefore \frac{dP}{dx} = 0$  when  $0.02x^2 = 202$

$$\begin{aligned} \therefore x^2 &= 10100 \\ \therefore x &= \sqrt{10100} \quad \{x > 0\} \\ \therefore x &\approx 100.49 \end{aligned}$$

and the sign diagram of  $\frac{dP}{dx}$  is:

$\therefore$  the maximum profit occurs when  $x \approx 100.49$

Now  $P(100) \approx \$938.63$  and  $P(101) \approx \$938.63$

$\therefore$  the maximum daily profit is  $\$938.63$  when 100 or 101 shirts are made.

- 4 a At time  $t = 0$ ,  $V = 20000e^{-0.4 \times 0}$   
 $= 20000$  dollars  
 $\therefore$  the purchase price of the car was  $\$20000$ .

b  $V' = -0.4(20000)e^{-0.4t}$   
 $= -8000e^{-0.4t}$

At time  $t = 10$ ,  $V' = -8000e^{-0.4 \times 10}$   
 $\approx -146.53$  dollars year $^{-1}$

$\therefore$  after 10 years, the car is decreasing in value at  $\$147$  per year.

5  $C(x) = 10 \ln x + \left(20 - \frac{x}{10}\right)^2 = 10 \ln x + 400 - 4x + \frac{x^2}{100}$

$\therefore C'(x) = \frac{10}{x} - 4 + \frac{x}{50} = \frac{500 - 200x + x^2}{50x}$

$\therefore C'(x) = 0$  when  $x^2 - 200x + 500 = 0$   
 $\therefore x = \frac{200 \pm \sqrt{38000}}{2}$   
 $\approx 2.53$  or  $197.47$

But  $x \geq 50$ , so  $x \approx 197.47$

Now  $C''(x) = -10x^{-2} + \frac{1}{50}$

$\therefore C''(197.47) = -10(197.47)^{-2} + 0.02 \approx 0.02$  which is  $> 0$ , so the shape is

$\therefore$  the minimum cost is when  $x \approx 197.47$

Sign diagram for  $C'(x)$ :

Now  $C(197) \approx 52.92$  and  $C(198) \approx 52.92$

$\therefore$  the manufacturer needs to produce 197 or 198 clocks per day to minimise costs.

6 a  $s(t) = 25t - 10 \ln t$  cm,  $t \geq 1$

$\therefore v(t) = 25 - \frac{10}{t}$  cm min $^{-1}$

$\therefore a(t) = 10t^{-2}$

$= \frac{10}{t^2}$  cm min $^{-2}$

b When  $t = e$ ,

$$\begin{aligned} s(e) &= 25e - 10 \ln e = 25e - 10 \text{ cm} \\ &\approx 58.0 \text{ cm} \end{aligned}$$

$$\begin{aligned} v(e) &= 25 - \frac{10}{e} \text{ cm min}^{-1} \\ &\approx 21.3 \text{ cm min}^{-1} \end{aligned}$$

$$a(e) = \frac{10}{e^2} \text{ cm min}^{-2} \approx 1.35 \text{ cm min}^{-2}$$

c As  $t \rightarrow \infty$ ,  $\frac{10}{t} \rightarrow 0$   $\therefore v(t) \rightarrow 25$  cm min $^{-1}$  (below)

d When  $v(t) = 20$  cm min $^{-1}$ ,

$$25 - \frac{10}{t} = 20$$

$$\therefore \frac{10}{t} = 5$$

$$\therefore t = 2 \text{ minutes}$$

### REVIEW SET 19C

1 a  $y = \ln(x^3 - 3x)$

$\therefore \frac{dy}{dx} = \frac{3x^2 - 3}{x^3 - 3x}$

b  $y = \frac{e^x}{x^2}$

$\therefore \frac{dy}{dx} = \frac{e^x(x^2) - e^x(2x)}{x^4}$  {quotient rule}  
 $= \frac{e^x(x - 2)}{x^3}$

2  $y = \ln(x^4 + 3)$

$\therefore \frac{dy}{dx} = \frac{4x^3}{x^4 + 3}$

$\therefore$  when  $x = 1$ ,  $\frac{dy}{dx} = \frac{4(1)^3}{1^4 + 3} = 1$  and  $y = \ln(1^4 + 3) = \ln 4$

$\therefore$  the tangent has equation  $\frac{y - \ln 4}{x - 1} = 1$  or  $y = x - 1 + \ln 4$

Now when  $x = 0$ ,  $y = \ln 4 - 1$ , so the tangent cuts the  $y$ -axis at  $(0, \ln 4 - 1)$ .

3 a  $e^{2x} = 3e^x$

$\therefore e^{2x} - 3e^x = 0$

$\therefore e^x(e^x - 3) = 0$

$\therefore e^x = 0$  or  $3$

$\therefore e^x = 3$  {as  $e^x > 0$ }

$\therefore x = \ln 3$

b  $e^{2x} - 7e^x + 12 = 0$

$\therefore (e^x - 3)(e^x - 4) = 0$

$\therefore e^x = 3$  or  $4$

$\therefore x = \ln 3$  or  $\ln 4$

4 a  $f(x) = e^x - x$

$\therefore f'(x) = e^x - 1$

so  $f'(x) = 0$  when  $e^x = 1$

$\therefore x = 0$

b As  $x \rightarrow \infty$ ,  $e^x \rightarrow \infty$  faster than  $x$

$\therefore f(x) \rightarrow \infty$

Sign diagram of  $f'(x)$  is:

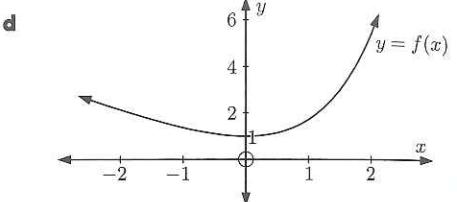
Now  $f(0) = e^0 - 0 = 1$

$\therefore$  there is a local minimum at  $(0, 1)$ .

c  $f''(x) = e^x$   
 $\therefore f''(x) > 0$  for all  $x$   
 $\therefore f(x)$  is concave up for all  $x$

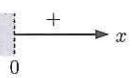
e Since a local minimum exists at  $(0, 1)$ ,  
 $f(x) \geq 1$  for all  $x$   
 $\therefore e^x - x \geq 1$   
 $\therefore e^x \geq x + 1$  for all  $x$

5 a  $f(x) = \ln(e^x + 3)$   
 $\therefore f'(x) = \frac{e^x}{e^x + 3}$



6 a  $f(x) = x + \ln x$  is defined when  $x > 0$

b  $f'(x) = 1 + \frac{1}{x} = \frac{x+1}{x}$  which has sign diagram:

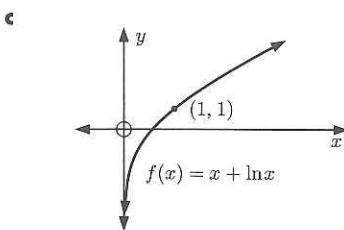


$\therefore f(x)$  is increasing for all  $x > 0$ .

f  $f''(x) = -\frac{1}{x^2}$  which has sign diagram:



$\therefore f(x)$  is concave down for all  $x > 0$ .



d  $f(1) = 1 + \ln(1) = 1$

$\therefore (1, 1)$  is the point of contact.

$f'(1) = \frac{1+1}{1} = 2$

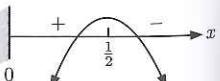
$\therefore$  the tangent at  $x = 1$  has gradient 2,  
so the normal has gradient  $-\frac{1}{2}$

$\therefore$  the normal has equation  $\frac{y-1}{x-1} = -\frac{1}{2}$   
 $\therefore 2y-2 = -x+1$   
 $\therefore x+2y=3$

7 Let the coordinates of B be  $(x, 0)$ , so the coordinates of A are  $(x, e^{-2x})$ .

$\therefore$  the area OBAC is  $A = xe^{-2x}$

$\therefore \frac{dA}{dx} = (1)e^{-2x} + x(-2e^{-2x})$  {product rule}  
 $= e^{-2x}(1-2x)$   
 $= \frac{1-2x}{e^{2x}}$  and has sign diagram:



So, the maximum area occurs when  $x = \frac{1}{2}$  and  $y = e^{-2(\frac{1}{2})} = e^{-1} = \frac{1}{e}$

$\therefore$  the coordinates of A are  $(\frac{1}{2}, \frac{1}{e})$ .

## Chapter 20

### DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

#### EXERCISE 20A

1 a  $y = \sin(2x)$   
 $\therefore \frac{dy}{dx} = \cos(2x) \frac{d}{dx}(2x)$   
 $= 2\cos(2x)$

c  $y = \cos(3x) - \sin x$   
 $\therefore \frac{dy}{dx} = -\sin(3x) \times 3 - \cos x$   
 $= -3\sin(3x) - \cos x$

e  $y = \cos(3-2x)$   
 $\therefore \frac{dy}{dx} = -\sin(3-2x) \times -2$   
 $= 2\sin(3-2x)$

g  $y = \sin\left(\frac{x}{2}\right) - 3\cos x$   
 $\therefore \frac{dy}{dx} = \frac{1}{2}\cos\left(\frac{x}{2}\right) + 3\sin x$

i  $y = 4\sin x - \cos(2x)$   
 $\therefore \frac{dy}{dx} = 4\cos x + \sin(2x) \times 2$   
 $= 4\cos x + 2\sin(2x)$

2 a  $y = x^2 + \cos x$   
 $\therefore \frac{dy}{dx} = 2x - \sin x$

c  $y = e^x \cos x$   
 $\therefore \frac{dy}{dx} = e^x \cos x + e^x(-\sin x)$   
 $= e^x \cos x - e^x \sin x$

e  $y = \ln(\sin x)$   
 $\therefore \frac{dy}{dx} = \frac{\cos x}{\sin x}$

g  $y = \sin(3x)$   
 $\therefore \frac{dy}{dx} = 3\cos(3x)$

b  $y = \sin x + \cos x$   
 $\therefore \frac{dy}{dx} = \cos x - \sin x$

d  $y = \sin(x+1)$   
 $\therefore \frac{dy}{dx} = \cos(x+1) \frac{d}{dx}(x+1)$   
 $= 1\cos(x+1)$   
 $= \cos(x+1)$

f  $y = \tan(5x)$   
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2(5x)} \times 5$   
 $= \frac{5}{\cos^2(5x)}$

h  $y = 3\tan(\pi x)$   
 $\therefore \frac{dy}{dx} = 3 \times \frac{1}{\cos^2(\pi x)} \times \pi$   
 $= \frac{3\pi}{\cos^2(\pi x)}$

b  $y = \tan x - 3\sin x$   
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} - 3\cos x$

d  $y = e^{-x} \sin x$   
 $\therefore \frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$

f  $y = e^{2x} \tan x$   
 $\therefore \frac{dy}{dx} = 2e^{2x} \tan x + \frac{e^{2x}}{\cos^2 x}$

h  $y = \cos(\frac{x}{2})$   
 $\therefore \frac{dy}{dx} = -\frac{1}{2}\sin(\frac{x}{2})$