Exponential and Logarithmic Functions

Chapter 31 Supplement

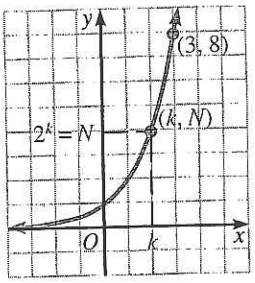
Pages 470 - 71	(1 – 38)
Page 476 - 77	(1 – 42, 45 – 51)
Pages 481 -82	(9 – 26)
Pages 490-91	(1 – 47)

Logarithmic Functions

10-4 Definition of Logarithms

Objective To define logarithmic functions and to learn how they are related to exponential functions.

The graph of $f(x) = 2^x$ is shown in Figure 1. Since the function passes the horizontal-line test, it has an inverse function. The graph of f^{-1} can be found by reflecting the graph of f across the line $y = x$, as shown in Figure 2.



$f(x) = 2^x$

Figure 1

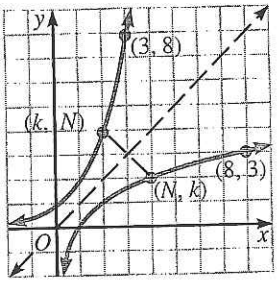
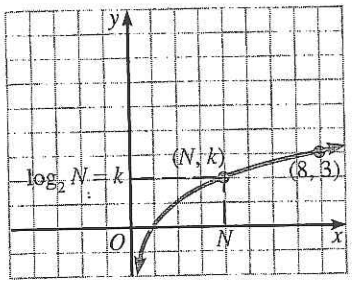


Figure 2



$f^{-1}(x) = \log_2 x$

Figure 3

The inverse of f, the exponential function with base 2, is called the **logarithmic function** with base 2, and is denoted by $f^{-1}(x) = \log_2 x$. (See Figure 3.)

$\log_2 x$ is read "the base 2 logarithm of x" or "log base 2 of x."

Notice that (3, 8) is on the graph of $f(x) = 2^x$ and (8, 3) is on the graph of $f^{-1}(x) = \log_2 x$. Since these functions are inverses, it follows that $2^3 = 8$ means $\log_2 8 = 3$. This and other examples are given below.

Exponential Form		*Logarithmic Form*
$2^3 = 8$	means	$\log_2 8 = 3$
$2^4 = 16$	means	$\log_2 16 = 4$
$2^0 = 1$	means	$\log_2 1 = 0$
$2^{-1} = \frac{1}{2}$	means	$\log_2 \frac{1}{2} = -1$
$2^k = N$	means	$\log_2 N = k$

A base other than 2 can be used with logarithmic functions. In fact the base can be any positive number except 1, since every power of 1 is 1.

Definition of Logarithm

If b and N are positive numbers ($b \neq 1$),

$$\log_b N = k \text{ if and only if } b^k = N.$$

Every positive number has a unique logarithm with base b. Since the base b exponential function is one-to-one, its inverse is one-to-one. Therefore,

$$\log_b M = \log_b N \text{ if and only if } M = N.$$

Several other properties of logarithms are easy to verify. Since the statements $\log_b N = k$ and $b^k = N$ are equivalent, by substitution

$$\log_b b^k = k \quad \text{and} \quad b^{\log_b N} = N.$$

Since $b = b^1$ and $1 = b^0$, it follows from the definition of a logarithm that

$$\log_b b = 1 \quad \text{and} \quad \log_b 1 = 0.$$

Example 1 Write each equation in exponential form.

a. $\log_6 36 = 2$ **b.** $\log_2 2 = 1$ **c.** $\log_{10} (0.001) = -3$

Solution **a.** $6^2 = 36$ **b.** $2^1 = 2$ **c.** $10^{-3} = 0.001$

Example 2 Write each equation in logarithmic form.

a. $6^0 = 1$ **b.** $8^{-2/3} = \frac{1}{4}$ **c.** $5^{3/2} = 5\sqrt{5}$

Solution **a.** $\log_6 1 = 0$ **b.** $\log_8 \frac{1}{4} = -\frac{2}{3}$ **c.** $\log_5 (5\sqrt{5}) = \frac{3}{2}$

Example 3 Simplify each logarithm.

a. $\log_5 25$ **b.** $\log_2 8\sqrt{2}$ **c.** $2^{\log_2 7}$

Solution **a.** Since $5^2 = 25$, $\log_5 25 = 2$. ***Answer***

b. Let $\log_2 8\sqrt{2} = x$.

Then:
$$2^x = 8\sqrt{2}$$
$$2^x = 2^3 \cdot 2^{1/2}$$
$$2^x = 2^{7/2} \qquad \text{Set the exponents equal.}$$
$$x = \frac{7}{2}$$

$\therefore \log_2 8\sqrt{2} = \frac{7}{2}$ ***Answer***

c. Since $b^{\log_b N} = N$, $2^{\log_2 7} = 7$. ***Answer***

Example 4 Solve each equation.

a. $\log_4 x = 3$

b. $\log_x 81 = 4$

Solution

a. Rewrite in exponential form.

$$4^3 = x$$
$$64 = x$$

$\therefore$ the solution set is $\{64\}$. *Answer*

b. Rewrite in exponential form.

$$x^4 = 81$$

Since $81 = 3^4$,

$$x^4 = 3^4$$
$$x = 3$$

$\therefore$ the solution set is $\{3\}$. *Answer*

Oral Exercises

Express in exponential form.

1. $\log_2 32 = 5$ **2.** $\log_3 9 = 2$ **3.** $\log_7 \sqrt{7} = \frac{1}{2}$ **4.** $\log_3 \frac{1}{81} = -4$

Express in logarithmic form.

5. $4^3 = 64$ **6.** $9^{3/2} = 27$ **7.** $10^{-2} = 0.01$ **8.** $16^{-3/4} = \frac{1}{8}$

Simplify.

9. $\log_6 36$ **10.** $\log_2 16$ **11.** $\log_{10} 100$ **12.** $\log_3 \frac{1}{9}$

13. $\log_2 2\sqrt{2}$ **14.** $\log_7 1$ **15.** $4^{\log_4 16}$ **16.** $\log_6 (6^5)$

17. $\log_2 10$ lies between the consecutive integers _?_ and _?_.

18. $\log_{10} 101$ lies between the consecutive integers _?_ and _?_.

19. $\log_3 \frac{1}{50}$ lies between the consecutive integers _?_ and _?_.

20. If you solve $\log_2 x^2 = \log_2 100$, then the solution set is $\{?, ?\}$.

Written Exercises

Simplify each logarithm.

A

1. $\log_5 125$ **2.** $\log_4 16$ **3.** $\log_3 81$

4. $\log_6 6$ **5.** $\log_3 1$ **6.** $\log_8 4$

7. $\log_5 \frac{1}{25}$ **8.** $\log_2 \frac{1}{8}$ **9.** $\log_6 6\sqrt{6}$

10. $\log_5 25\sqrt{5}$ **11.** $\log_4 \sqrt{2}$ **12.** $\log_{27} \sqrt{3}$

13. $\log_7 \sqrt[3]{49}$

14. $\log_3 \sqrt[5]{9}$

15. $\log_{1/2} 8$

16. $\log_{1/3} 27$

17. $\log_2 \sqrt[3]{\frac{1}{4}}$

18. $\log_{10} \frac{1}{\sqrt{1000}}$

Solve for x.

19. $\log_7 x = 2$

20. $\log_6 x = 3$

21. $\log_9 x = -\frac{1}{2}$

22. $\log_6 x = 2.5$

23. $\log_4 x = -\frac{3}{2}$

24. $\log_{1/9} x = -\frac{1}{2}$

B 25. $\log_x 27 = \frac{3}{2}$

26. $\log_x 64 = 6$

27. $\log_x 7 = -\frac{1}{2}$

28. $\log_x 7 = 1$

29. $\log_x 1 = 0$

30. $\log_x 2 = 0$

31. a. Show that $\log_2 8 + \log_2 4 = \log_2 32$ by simplifying the three logarithms.
    b. Show that $\log_9 3 + \log_9 27 = \log_9 81$ by simplifying the three logarithms.
    c. State a generalization based on parts (a) and (b).

32. a. Simplify $\log_2 8$ and $\log_8 2$.
    b. Simplify $\log_3 \sqrt{3}$ and $\log_{\sqrt{3}} 3$.
    c. State a generalization based on parts (a) and (b).

33. a. If $f(x) = 6^x$, then $f^{-1}(x) =$ _?_.
    b. Find $f^{-1}(36)$ and $f^{-1}\left(\frac{1}{\sqrt{6}}\right)$.
    c. Give the domain and range of f and f^{-1}.

34. a. If $g(x) = \log_4 x$, then $g^{-1}(x) =$ _?_.
    b. Find $g^{-1}(2)$ and $g^{-1}\left(-\frac{3}{2}\right)$.
    c. Give the domain and range of g and g^{-1}.

Sketch the graph of each function and its inverse function in the same coordinate system. Label at least three points on each graph. You may wish to check your graphs on a computer or a graphing calculator.

35. $f(x) = 6^x$

36. $g(x) = \log_4 x$

37. $h(x) = \log_{10} x$

38. $k(x) = \left(\frac{1}{2}\right)^x$

Solve.

C 39. $\log_5 (\log_3 x) = 0$

40. $\log_4 (\log_3 (\log_2 x)) = 0$

41. If $0 < b < 1$ and $0 < u < 1$, is $\log_b u$ positive or negative?

42. If $1 < a < b$, is $\log_b (\log_b a)$ positive or negative?

10-5 Laws of Logarithms

Objective To learn and apply the basic properties of logarithms.

The laws of exponents can be used to derive the *laws of logarithms*.

Laws of Logarithms

Let b be the base of a logarithmic function ($b > 0$, $b \neq 1$). Let M and N be positive numbers.

1. $\log_b MN = \log_b M + \log_b N$
2. $\log_b \frac{M}{N} = \log_b M - \log_b N$
3. $\log_b M^k = k \log_b M$

Laws 1 and 3 are proved below. The proof of Law 2 is left as Exercise 43.

Proof: Let $\log_b M = x$ and $\log_b N = y$. Therefore, $b^x = M$ and $b^y = N$.

Law 1	*Law 3*
$MN = b^x b^y = b^{x+y}$	$M^k = (b^x)^k = b^{kx}$
$\log_b MN = \log_b b^{x+y}$	$\log_b M^k = \log_b b^{kx}$
$\log_b MN = x + y$	$\log_b M^k = kx$
$\therefore \log_b MN = \log_b M + \log_b N$	$\therefore \log_b M^k = k \log_b M$

Example 1 Express $\log_6 M^2N^3$ in terms of $\log_6 M$ and $\log_6 N$.

Solution

$$\log_6 M^2N^3 = \log_6 M^2 + \log_6 N^3 \quad \text{Use Law 1.}$$
$$= 2 \log_6 M + 3 \log_6 N \quad \text{Use Law 3.}$$

Example 2 Express $\log_2 \sqrt{\frac{M}{N^5}}$ in terms of $\log_2 M$ and $\log_2 N$.

Solution

$$\log_2 \sqrt{\frac{M}{N^5}} = \log_2 \left(\frac{M}{N^5}\right)^{1/2}$$
$$= \frac{1}{2} \log_2 \left(\frac{M}{N^5}\right) \quad \text{Use Law 3.}$$
$$= \frac{1}{2}(\log_2 M - \log_2 N^5) \quad \text{Use Law 2.}$$
$$= \frac{1}{2}(\log_2 M - 5 \log_2 N) \quad \text{Use Law 3.}$$

The process illustrated in Examples 1 and 2 can also be reversed.

Example 3 Express as a single logarithm.

a. $\log_{10} p + 3 \log_{10} q$ **b.** $4 \log_{10} p - 2 \log_{10} q$

Solution

a. $\log_{10} p + 3 \log_{10} q = \log_{10} p + \log_{10} q^3$ Use Law 3.

$= \log_{10} pq^3$ Use Law 1.

b. $4 \log_{10} p - 2 \log_{10} q = \log_{10} p^4 - \log_{10} q^2$ Use Law 3.

$= \log_{10} \dfrac{p^4}{q^2}$ Use Law 2.

You can often use the laws of logarithms to compute the numerical value of a logarithm from given logarithms.

Example 4 If $\log_{10} 2 = 0.30$ and $\log_{10} 3 = 0.48$, find the following.

a. $\log_{10} 18$ **b.** $\log_{10} \dfrac{20}{3}$ **c.** $\log_{10} 5$ **d.** $\log_{10} \left(\dfrac{1}{\sqrt[3]{2}}\right)$

Solution

a. Since $18 = 2 \cdot 3^2$,

$$\begin{aligned} \log_{10} 18 &= \log_{10} 2 + 2 \log_{10} 3 \\ &= 0.30 + 2(0.48) \\ &= 1.26 \quad \textit{Answer} \end{aligned}$$

b. Since $\dfrac{20}{3} = \dfrac{2 \cdot 10}{3}$,

$$\begin{aligned} \log_{10} \frac{20}{3} &= \log_{10} 2 + \log_{10} 10 - \log_{10} 3 \\ &= 0.30 + 1 - 0.48 \\ &= 0.82 \quad \textit{Answer} \end{aligned}$$

c. Since $5 = \dfrac{10}{2}$,

$$\begin{aligned} \log_{10} 5 &= \log_{10} 10 - \log_{10} 2 \\ &= 1 - 0.30 \\ &= 0.70 \quad \textit{Answer} \end{aligned}$$

d. Since $\dfrac{1}{\sqrt[3]{2}} = 2^{-1/3}$,

$$\begin{aligned} \log_{10} \left(\frac{1}{\sqrt[3]{2}}\right) &= -\frac{1}{3} \log_{10} 2 \\ &= -\frac{1}{3}(0.30) \\ &= -0.10 \quad \textit{Answer} \end{aligned}$$

You can use the laws and properties of logarithms to solve logarithmic equations. *Because the logarithm of a variable expression is defined only if the expression is positive, be sure to check the answers you obtain.*

Example 5 Solve $\log_3 x + \log_3 (x - 6) = 3$ for x.

Solution

$$\log_3 x + \log_3 (x - 6) = 3$$

$$\log_3 x(x - 6) = 3 \qquad \text{Use Law 1.}$$

$$\log_3 (x^2 - 6x) = 3$$

Change to exponential form: $x^2 - 6x = 3^3$

$$x^2 - 6x - 27 = 0$$

$$(x + 3)(x - 9) = 0$$

$$x = -3 \quad \text{or} \quad x = 9$$

Check: If $x = -3$, $\log_3 x$ is not defined. So -3 is *not* a root.

If $x = 9$, $\log_3 x + \log_3 (x - 6) = \log_3 9 + \log_3 3$

$$= \log_3 (9 \cdot 3)$$

$$= \log_3 27$$

$$= 3 \ \checkmark$$

$\therefore$ the solution set is $\{9\}$. ***Answer***

Oral Exercises

Express each logarithm in terms of $\log_3 M$ and $\log_3 N$.

1. $\log_3 M^4$
2. $\log_3 N^6$
3. $\log_3 M^4N$
4. $\log_3 \left(\frac{M}{N^3}\right)$
5. $\log_3 \left(\frac{1}{M}\right)$
6. $\log_3 \sqrt{M}$
7. $\log_3 \sqrt[3]{N^2}$
8. $\log_3 \left(\frac{1}{N\sqrt{N}}\right)$

Express as a logarithm of a single number or expression.

9. $\log_a 3 + \log_a 4$
10. $\log_a 7 - \log_a 5$
11. $4 \log_a 2$
12. $2 \log_a 9$
13. $\frac{1}{2} \log_a 36$
14. $-\log_a \frac{1}{6}$
15. $\log_b 3 + \log_b 5 + \log_b 2$
16. $\log_b 6 + \log_b 5 - \log_b 2$
17. $2 \log_b p + \log_b q$
18. $\log_b x - 3 \log_b y$
19. $\frac{1}{2} \log_b r + \frac{1}{2} \log_b s$
20. $\frac{1}{2}(\log_b x - \log_b y)$

Let $c = \log_3 10$ and $d = \log_3 5$. Express the following in terms of c and d.

21. $\log_3 50$
22. $\log_3 500$
23. $\log_3 250$
24. $\log_3 2$

Written Exercises

Express each logarithm in terms of $\log_2 M$ and $\log_2 N$.

A

1. $\log_2 M^6N^3$
2. $\log_2 (MN)^4$
3. $\log_2 M\sqrt{N}$
4. $\log_2 \sqrt[3]{M^2N}$
5. $\log_2 \frac{M^4}{N^3}$
6. $\log_2 \left(\frac{M}{N}\right)^7$
7. $\log_2 \sqrt{\frac{M}{N^3}}$
8. $\log_2 \frac{1}{MN}$

If $\log_{10} 9 = 0.95$ and $\log_{10} 2 = 0.30$ (accurate to two decimal places), find the following.

9. $\log_{10} 81$
10. $\log_{10} \frac{9}{2}$
11. $\log_{10} \sqrt{2}$
12. $\log_{10} 3$
13. $\log_{10} 8$
14. $\log_{10} 36$
15. $\log_{10} \frac{20}{9}$
16. $\log_{10} 900$
17. $\log_{10} \frac{1}{9}$
18. $\log_{10} \frac{1}{2000}$
19. $\log_{10} \sqrt[3]{\frac{2}{9}}$
20. $\log_{10} 162$

Express as a logarithm of a single number or expression.

21. $5 \log_4 p + \log_4 q$
22. $\log_{10} x - 4 \log_{10} y$
23. $4 \log_3 A - \frac{1}{2} \log_3 B$
24. $\log_5 M + \frac{1}{4} \log_5 N$

B

25. $\log_2 M + \log_2 N + 3$
26. $\log_5 x - \log_5 y + 2$
27. $1 - 3 \log_5 x$
28. $\frac{1 + \log_9 x}{2}$

Simplify.

29. $2 \log_{10} 5 + \log_{10} 4$
30. $2 \log_3 6 - \log_3 4$
31. $\log_4 40 - \log_4 5$
32. $\log_4 3 - \log_4 48$

Solve each equation.

33. $\log_a x = 2 \log_a 3 + \log_a 5$
34. $\log_a x = \frac{3}{2} \log_a 9 + \log_a 2$
35. $\log_b (x + 3) = \log_b 8 - \log_b 2$
36. $\log_b (x^2 + 7) = \frac{2}{3} \log_b 64$
37. $\log_a x - \log_a (x - 5) = \log_a 6$
38. $\log_a (3x + 5) - \log_a (x - 5) = \log_a 8$
39. $\log_2 (x^2 - 9) = 4$
40. $\log_3 (x + 2) + \log_3 6 = 3$
41. If $f(x) = \log_2 x$ and $g(x) = 4^x$, find: **a.** $f(g(3))$ **b.** $g\left(f\left(\frac{1}{2}\right)\right)$ **c.** $f(g^{-1}(16))$
42. If $f(x) = 3^x$ and $g(x) = \log_9 x$, find: **a.** $f(f(-1))$ **b.** $g\left(g\left(\frac{1}{81}\right)\right)$ **c.** $f^{-1}(g(9))$
43. Prove the second law of logarithms.
44. Simplify $4^{\log_2 (2^{\log_2 5})}$.

Solve each equation.

C **45.** $\log_5 (\log_3 x) = 0$

46. $\log_2 (\log_4 x) = 1$

47. $\log_6 (x + 1) + \log_6 x = 1$

48. $\log_{10} (x + 6) + \log_{10} (x - 6) = 2$

49. $\frac{1}{2} \log_a (x + 2) + \frac{1}{2} \log_a (x - 2) = \frac{2}{3} \log_a 27$

50. $2 \log_3 x - \log_3 (x - 2) = 2$

51. $\log_b (x - 1) + \log_b (x + 2) = \log_b (8 - 2x)$

Mixed Review Exercises

Solve.

1. $\log_2 x = -\frac{1}{2}$

2. $x^3 - 7x + 6 = 0$

3. $2^{x+3} = 4^{x-1}$

4. $\sqrt{x} + 2 = x$

5. $3^x = \frac{\sqrt{3}}{9}$

6. $3(2x - 1) = 5x + 4$

7. $\frac{x}{x+1} - 1 = \frac{1}{x}$

8. $\log_x 8 = \frac{3}{2}$

9. $(x - 2)^2 = 5$

If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x + 1}$, find each of the following.

10. $f(-2)$

11. $g(3)$

12. $f(g(1))$

13. $g(f(-1))$

Self-Test 2

Vocabulary logarithmic function (p. 468)
logarithm (p. 469)
laws of logarithms (p. 473)

1. Write in exponential form. **Obj. 10-4 p. 468**

a. $\log_3 81 = 4$

b. $\log_6 216 = 3$

2. Write in logarithmic form.

a. $5^4 = 625$

b. $25^{3/2} = 125$

3. Evaluate each expression.

a. $\log_2 4^{3/2}$

b. $4^{\log_4 12}$

4. Solve $\log_b 27 = 3$.

5. Write $\log_2 (M^5N^6)^{1/3}$ in terms of $\log_2 M$ and $\log_2 N$. **Obj. 10-5 p. 473**

6. If $\log_{10} 5 = 0.70$, find $\log_{10} 0.04$.

7. Solve $\log_a x + \log_a (x - 2) = \log_a 3$.

Check your answers with those at the back of the book.

Applications

10-6 Applications of Logarithms

Objective To use common logarithms to solve equations involving powers and to evaluate logarithms with any given base.

Logarithms were invented to simplify difficult calculations. Because of the decimal nature of our number system, it is easiest to work with base 10 logarithms. These are called **common logarithms.** When common logarithms are used in calculations, the base 10 is usually not written. For example, log 6 means $\log_{10} 6$.

Most scientific calculators have a key marked "log" that gives the common logarithm of a number. You also can find the common logarithm of a number by using Table 3 on page 812.

Although calculators and computers have replaced logarithms for doing heavy computational work, logarithms still provide the best, or even the only, method of solution for many types of problems. Some of these will be shown later in this lesson and in the next.

If you have a calculator, you should practice finding the logarithms of a few numbers and then skip ahead to the paragraph preceding Example 2.

Without a calculator, you will need to use the table of logarithms (Table 3), a portion of which is shown below.

N	0	1	2	3	4	5	6	7	8	9
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962

The table gives the logarithms of numbers between 1 and 10 rounded to four decimal places. The decimal point is omitted. To find an approximation for $\log x$ for $1 < x < 10$, find the first two digits of x in the column headed N and the third digit in the row to the right of N. For example, to find log 2.36, look for 23 under N and move across row 23 to the column headed 6, where you find 3729. Therefore, $\log 2.36 \approx 0.3729$.

To find an approximation for the logarithm of a positive number greater than 10 or less than 1, write the number in scientific notation (see page 221) and use Law 1: $\log MN = \log M + \log N$. For example,

$$\begin{aligned} \log 236 &= \log (2.36 \times 10^2) \\ &= \log 2.36 + \log 10^2 && \text{Use Law 1.} \\ &= \log 2.36 + 2 \\ &\approx 0.3729 + 2, \text{ or } 2.3729 && \text{Use Table 3.} \end{aligned}$$

The example shows that the common logarithm of a number can be written as the sum of an integer, called the **characteristic,** and a nonnegative number less than 1, called the **mantissa,** which can be found in a table of logarithms.

Example 1 Find each logarithm. Use Table 3.

a. log 3.8 **b.** log 97,500 **c.** log 0.000542

Solution **a.** Find 38 under N; then read across to the column headed 0.

$\therefore$ $\log 3.8 = 0.5798$ ***Answer***

b. Write 97,500 in scientific notation and use laws of logarithms.

$$\log 97{,}500 = \log (9.75 \times 10^4) = \log 9.75 + \log 10^4$$

The mantissa for 9.75 is 0.9890. The characteristic is 4.

$\therefore$ $\log 97{,}500 = 0.9890 + 4 = 4.9890$ ***Answer***

c. Write 0.000542 in scientific notation and use laws of logarithms.

$$\log 0.000542 = \log (5.42 \times 10^{-4}) = \log 5.42 + \log 10^{-4}$$

The mantissa for 5.42 is 0.7340. The characteristic is -4.

$\therefore$ $\log 0.000542 = 0.7340 + (-4) = -3.2660$ ***Answer***

The logarithms in the tables, and therefore the answers, are approximations. However, it is a common practice to use $=$, as in Example 1, rather than $\approx$.

If $\log y = a$, then the number y is sometimes called the **antilogarithm** of a. The value of y can be found by using a calculator or log tables.

Example 2 Find y to three significant digits if:

a. $\log y = 0.8995$ **b.** $\log y = 2.4825$

Solution 1 Using a Calculator On some calculators you find antilogarithms by using the inverse function key with the logarithmic function key. On many others, you can use the 10^x key as shown.

a. If $\log_{10} y = 0.8995$, then $y = 10^{0.8995} = 7.93$. ***Answer***

b. $y = 10^{2.4825} = 304$ ***Answer***

Solution 2 Using Tables

a. In the body of Table 3 find the value closest to 0.8995, that is, 0.8993. This is the entry in the row labeled 79 and in the column labeled 3.

$\therefore y = 7.93$ ***Answer***

b. First find the number that has 0.4825 as its mantissa. The table entry with mantissa closest to 0.4825 is 3.04.

$$\begin{aligned} \log y = 2.4825 &= 0.4825 + 2 \\ &= \log 3.04 + \log 10^2 \\ &= \log (3.04 \times 10^2) \\ \log y &= \log 304 \end{aligned}$$

$\therefore y = 304$ ***Answer***

In the previous example the answers were rounded to three significant digits. For most problems the answers you find using Table 3 will be accurate to three significant digits.

Example 3 Find the value of each expression to three significant digits.

a. $\sqrt[5]{493}$ **b.** $(0.173)^6$

Solution 1 Using a Calculator

a. Since $\sqrt[5]{493} = 493^{1/5} = 493^{0.2}$, use the power key, y^x.

$y^x = 493^{0.2} = 3.46$ ***Answer***

b. $y^x = (0.173)^6 = 0.0000268$ ***Answer***

Solution 2 Using Tables

a. Let $x = \sqrt[5]{493}$.

$\log x = \frac{1}{5} \log 493$

$\log x = \frac{1}{5}(2.6928)$

$\log x = 0.5386$

$\therefore \quad x = 3.46$ ***Answer***

b. $(0.173)^6 = (1.73 \times 10^{-1})^6$
$= (1.73)^6 \times 10^{-6}$

Let $x = (1.73)^6$. Then:

$\log x = 6 \log 1.73$

$\log x = 1.428$

$x = 26.8$

$\therefore (0.173)^6 = 26.8 \times 10^{-6}$
$= 0.0000268$ ***Answer***

Whether or not you use a calculator, you need the properties of logarithms to solve the exponential equation given in Example 4. The solution to Example 4 is given in *calculation-ready form,* where the next step is to obtain a decimal approximation using a calculator or a table.

Example 4 Solve $3^{2x} = 5$.

a. Give the solution in calculation-ready form.

b. Give the solution as a decimal with three significant digits.

Solution **a.**

$3^{2x} = 5$ Take logarithms of both sides.

$\log 3^{2x} = \log 5$

$2x \log 3 = \log 5$ Use laws of logarithms to simplify.

$x = \dfrac{\log 5}{2 \log 3}$ Solve for x.

$\therefore$ the solution in calculation-ready form is $\dfrac{\log 5}{2 \log 3}$. ***Answer***

b. Find log 5 and log 3 with a calculator or table.

$$x = \frac{0.6990}{2(0.4771)} = 0.733$$

$\therefore$ to three significant digits, the solution is 0.733. ***Answer***

If you know the base b logarithm of a number and wish to find its base a logarithm, you can use the following formula:

Change-of-Base Formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The proof of this formula is left as Exercise 42 on page 482.

Example 5 Find $\log_4 7$.

Solution Use the change-of-base formula letting $b = 10$:

$$\log_4 7 = \frac{\log_{10} 7}{\log_{10} 4} = \frac{0.8451}{0.6021} = 1.404 \quad \textit{Answer}$$

Oral Exercises

Use a calculator or Table 3 to find each logarithm.

1. $\log 4.05$
2. $\log 40.5$
3. $\log 405$
4. $\log 0.405$
5. $\log 8.36$
6. $\log 83.6$
7. $\log 0.836$
8. $\log 0.0836$

Use a calculator or Table 3 to find x to three significant digits.

9. $\log x = 0.6072$
10. $\log x = 0.9212$
11. $\log x = 1.9212$
12. $\log x = 0.9212 - 1$

Solve for x in calculation-ready form.

13. $2^x = 7$
14. $9^x = 8$
15. $8^{2x} = 3$
16. $3^{-x} = 7$

Written Exercises

Use a calculator or Table 3 to find the value of each expression to three significant digits.

A

1. $(1.06)^{10}$
2. $(10.6)^{10}$
3. $(0.38)^5$
4. $(347)^{1.5}$
5. $(12.7)^{5/2}$
6. $\sqrt[6]{786}$
7. $\sqrt[5]{(81.2)^4}$
8. $\sqrt[3]{(412)^2}$

Use a calculator or Table 3 to find x to three significant digits.

9. $\log x = 0.8531$
10. $\log x = 0.4065$
11. $\log x = 2.84$
12. $\log x = 1.605$
13. $\log x = -1.8$
14. $\log x = -2.91$

Solve each equation.

a. Give the solution in calculation-ready form.
b. Give the solution to three significant digits.

15. $3^x = 30$ **16.** $5^t = 10$ **17.** $5.6^x = 56$ **18.** $(1.02)^x = 2$

19. $30^{-x} = 5$ **20.** $12^{2x} = 1000$ **21.** $3.5^{2t} = 60$ **22.** $\frac{4^{2-t}}{3} = 7$

Solve each equation *without* using a calculator or logarithms. (See Example 2 of Lesson 10-2, page 461.)

23. $4^x = 8\sqrt{2}$ **24.** $3^x = \sqrt[5]{9}$ **25.** $125^x = 25\sqrt{5}$ **26.** $8^x = 16\sqrt[3]{2}$

Unlike Exercises 15–26, Exercises 27–34 are *not* exponential equations. Solve each equation using a calculator or logarithms. Give answers to three significant digits.

Sample Solve $x^{2/3} = 12$.

Solution 1. Raise each side to the $\frac{3}{2}$ power. You find that $x = 12^{3/2}$.

2. To simplify $12^{3/2}$.
   (a) Use a calculator and the y^x key to obtain $x = 41.6$, or
   (b) Use logarithms to obtain $\log x = \frac{3}{2} \log 12 = 1.619$. Then find the antilogarithm: $x = 41.6$. *Answer*

27. $x^{2/5} = 34$ **28.** $x^{2/3} = 50$ **29.** $\sqrt[3]{x^4} = 60$ **30.** $\sqrt[5]{x^3} = 900$

B **31.** $2x^5 = 100$ **32.** $\frac{\sqrt[5]{x}}{9} = 7$ **33.** $(3y - 1)^6 = 80$ **34.** $\sqrt[3]{4t + 3} = 8.15$

Find each logarithm. Use the change-of-base formula.

35. $\log_2 9$ **36.** $\log_6 8$ **37.** $\log_3 40$ **38.** $\log_7 \frac{1}{2}$

Solve for x to three significant digits.

39. $3^{2x} - 7 \cdot 3^x + 10 = 0$ **40.** $3^{2x} - 7 \cdot 3^x + 12 = 0$

41. a. Simplify $\log_7 49$ and $\log_{49} 7$.
b. Simplify $\log_2 8$ and $\log_8 2$.
c. How are $\log_b a$ and $\log_a b$ related? Give a convincing argument to justify your answer.

C **42.** Derive the change-of-base formula. (*Hint:* Let $\log_a x = y$, so that $x = a^y$.)

10-8 The Natural Logarithm Function

Objective To define and use the natural logarithm function.

In advanced work in science and mathematics, the most important logarithm function is the **natural logarithm function.** Its base is the irrational number e, which has the approximate value 2.71828. The **natural logarithm** of x is sometimes denoted by $\log_e x$, but more often by $\ln x$.

The number e is defined to be the limiting value of $\left(1 + \frac{1}{n}\right)^n$ as n becomes larger and larger. Using a calculator, you can make a table and show that as n gets very large, the value of this expression approaches a number a bit larger than 2.718.

n	100	1000	10,000	100,000
$\left(1 + \frac{1}{n}\right)^n$	2.70481	2.71692	2.71815	2.71827

The exercises of this lesson are just like exercises in previous lessons, except that the symbol $\ln x$ is used instead of $\log_b x$. Example 1 illustrates.

Example 1

Working with base 2 logs

1. If $\log_2 x = 5$, then $x = 2^5$.
2. If $2^x = 7$, then $x = \log_2 7$.
3. $\log_2 2^5 = 5$ and $2^{\log_2 7} = 7$

Working with base e logs

1. If $\ln x = 5$, then $x = e^5$.
2. If $e^x = 7$, then $x = \ln 7$.
3. $\ln e^5 = 5$ and $e^{\ln 7} = 7$

Example 2 **a.** Simplify $\ln \frac{1}{e^2}$.

b. Write as a single logarithm: $2 \ln 5 + \ln 4 - 3$.

Solution **a.** $\ln \frac{1}{e^2} = \ln e^{-2} = -2 \ln e = -2 \cdot 1 = -2$ *Answer*

b. $2 \ln 5 + \ln 4 - 3 = \ln 5^2 + \ln 4 - \ln e^3$

$= \ln \frac{5^2 \cdot 4}{e^3} = \ln \frac{100}{e^3}$ *Answer*

Example 3 Solve: **a.** $\ln x = 2$ **b.** $\ln \frac{1}{x} = 2$

Solution **a.** $x = e^2$ *Answer*

b. $\frac{1}{x} = e^2$

$x = \frac{1}{e^2}$, or e^{-2} *Answer*

Example 4 Solve $e^{2x} = 9$.

Solution 1

$$e^{2x} = 9 \qquad \text{Rewrite in logarithmic form.}$$
$$2x = \ln 9$$
$$x = \frac{1}{2}\ln 9$$
$$x = \ln 9^{1/2} = \ln 3 \quad \textit{Answer}$$

Solution 2

$$e^{2x} = 9 \qquad \left\{\text{Take the natural log of both sides of the equation.}\right.$$
$$\ln e^{2x} = \ln 9$$
$$2x \cdot \ln e = \ln 9$$
$$2x \cdot 1 = \ln 9$$
$$x = \frac{1}{2}\ln 9 = \ln 9^{1/2} = \ln 3 \quad \textit{Answer}$$

Oral Exercises

Give each equation in exponential form.

1. $\ln 4 = 1.39$ **2.** $\ln \frac{1}{4} = -1.39$ **3.** $\ln e = 1$

Give each equation in logarithmic form.

4. $e^2 = 7.39$ **5.** $e^{-2} = 0.14$ **6.** $e^{1/5} = 1.22$

Simplify.

7. $\ln \frac{1}{e}$ **8.** $\ln e^{12}$ **9.** $\ln \sqrt{e}$

Solve.

10. $\ln x = 5$ **11.** $\ln x = \frac{1}{3}$ **12.** $e^x = 3$

13. $x = \ln e^{3/4}$ **14.** $e^{\ln x} = 10$ **15.** $\ln x = -\frac{1}{2}$

16. Approximate to three decimal places the value of $\left(1 + \frac{1}{5000}\right)^{5000}$.

Written Exercises

Write each equation in exponential form.

A **1.** $\ln 8 = 2.08$ **2.** $\ln 100 = 4.61$ **3.** $\ln \frac{1}{3} = -1.10$ **4.** $\ln \frac{1}{e^2} = -2$

Write each equation in logarithmic form.

5. $e^3 = 20.1$ 6. $e^7 = 1097$ 7. $e^{1/2} = 1.65$ 8. $\sqrt[3]{e} = 1.40$

Simplify. If the expression is undefined, say so.

9. $\ln e^2$ 10. $\ln e^{10}$ 11. $\ln \frac{1}{e^3}$ 12. $\ln \frac{1}{\sqrt{e}}$

13. $\ln 1$ 14. $\ln 0$ 15. $e^{\ln 5}$ 16. $e^{\ln 0.5}$

Write as a single logarithm.

17. $\ln 3 + \ln 4$ 18. $\ln 8 - \ln 2$ 19. $2 \ln 3 - \ln 5$

20. $\ln 7 + \frac{1}{2} \ln 9$ 21. $\frac{1}{3} \ln 8 + \ln 5 + 3$ 22. $4 \ln 2 - \ln 3 - 1$

Solve for x. Leave answers in terms of e.

23. $\ln x = 3$ 24. $\ln \frac{1}{x} = 2$ 25. $\ln (x - 4) = -1$

26. $\ln |x| = 1$ 27. $\ln x^2 = 9$ 28. $\ln \sqrt{x} = 3$

Solve for x. Leave answers in terms of natural logarithms.

29. $e^x = 2$ 30. $e^{-x} = 3$ 31. $e^{2x} = 25$

32. $e^{3x} = 8$ 33. $e^{x-2} = 2$ 34. $\frac{1}{e^x} = 7$

Solve. Leave answers in terms of e or natural logarithms.

35. $\sqrt{e^x} = 3$ 36. $e^{-2x} = 0.2$ 37. $(e^x)^5 = 1000$

38. $3e^{2x} + 2 = 50$ 39. $\ln (\ln x) = 0$ 40. $|\ln x| = 1$

41. $\ln x + \ln (x + 3) = \ln 10$ 42. $2 \ln x = \ln (x + 1)$ 43. $e^{2x} - 7e^x + 12 = 0$

Give the domain and range of each function.

B 44. $f(x) = \ln x$ 45. $f(x) = \ln |x|$ 46. $f(x) = \ln x^2$ 47. $f(x) = \ln (x - 5)$

48. Graph $y = \ln x$ and $y = e^x$ in the same coordinate system.

49. Graph $y = 2^x$, $y = e^x$, and $y = 3^x$ in the same coordinate system.

C 50. Express in terms of e the approximate value of each expression when n is very large.

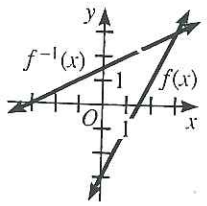
a. $\left(1 + \frac{1}{n}\right)^{5n}$ b. $\left(1 + \frac{2}{n}\right)^{n}$ c. $\left(\frac{n}{n+1}\right)^{2n}$

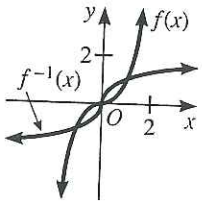
51. Refer to the compound interest formula on page 483.

a. Show that if interest is compounded daily, so that n is quite large, then $A \approx Pe^{rt}$.

b. Use part (a) to find the amount of money that you would have after 1 year if you invest \$1000 at 6% interest compounded daily.

Written Exercises, pages 466–467 **1. a.** $\frac{5}{2}$ **b.** -4 **c.** $-\frac{3}{2}$ **d.** $\frac{x-3}{2}$ **3. a.** $\frac{3}{2}$ **b.** 1 **c.** not real **d.** $\frac{\sqrt{x}}{2}$ **5. a.** 3 **b.** not real **c.** $\sqrt{x-3}$ **d.** $\sqrt[4]{x}$ **7.** no **9.** no **11.** $f^{-1}(x) = \frac{x+3}{2}$ **13.** $f^{-1}(x) = \sqrt[3]{x}$





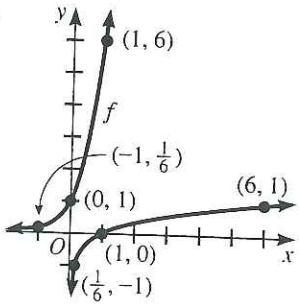
15. $g^{-1}(x) = \frac{8}{\sqrt[3]{x}}$ **17.** no inverse **19.** no inverse **21.** no inverse **23. c.** 1; 2; 3; -1 **d.** f: $D = \{\text{reals}\}$, $R = \{y: y > 0\}$; f^{-1}: $D = \{x: x > 0\}$, $R = \{\text{reals}\}$ **25.** $m = 1$, $b = 0$; or $m = -1$

Mixed Review Exercises, page 467 **1.** $\frac{1}{125}$ **2.** $7\sqrt{2}$ **3.** $-5 - 12i$ **4.** 81 **5.** -24 **6.** $\frac{1}{16}$ **7.** 9 **8.** $5 - 4\sqrt{2}$ **9.** -3 **10.** 128 **11.** $1 + 2i$ **12.** 2

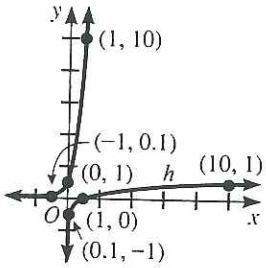
Self-Test 1, page 467 **1. a.** $2x^{2/3}y^{-1/3}$ **b.** $6^{-1/3}$ **2. a.** $\frac{25\sqrt{5}}{2}$ **b.** $x^2y\sqrt[6]{xy^5}$ **3.** $\{34\}$ **4. a.** $2^{-5\sqrt{2}}$ **b.** $2^{-8\sqrt{5}}$ **5.** $\{3\}$ **6. a.** 13 **b.** 5 **c.** $6\sqrt{x} + 1$ **d.** $\sqrt{6x+1}$ **7.** $f(g(x)) = 3\left(\frac{x+7}{3}\right) - 7 = x + 7 - 7 = x$; $g(f(x)) = \frac{(3x-7)+7}{3} = \frac{3x}{3} = x$

Written Exercises, pages 470–472 **1.** 3 **3.** 4 **5.** 0 **7.** -2 **9.** $\frac{3}{2}$ **11.** $\frac{1}{4}$ **13.** $\frac{2}{3}$ **15.** -3 **17.** $-\frac{2}{3}$ **19.** $\{49\}$ **21.** $\left\{\frac{1}{3}\right\}$ **23.** $\left\{\frac{1}{8}\right\}$ **25.** $\{9\}$ **27.** $\left\{\frac{1}{49}\right\}$ **29.** $\{x: x > 0 \text{ and } x \neq 1\}$ **31. a.** $3 + 2 = 5$ **b.** $\frac{1}{2} + \frac{3}{2} = 2$ **c.** $\log_b M + \log_b N = \log_b MN$ **33. a.** $\log_6 x$ **b.** 2; $-\frac{1}{2}$ **c.** f: $D = \{\text{reals}\}$, $R = \{y: y > 0\}$; f^{-1}: $D = \{x: x > 0\}$, $R = \{\text{reals}\}$

35.



37.



39. $\{3\}$
41. positive
43. a. 120 dB
b. 10^4

Written Exercises, pages 476–477
1. $6 \log_2 M + 3 \log_2 N$ **3.** $\log_2 M + \frac{1}{2} \log_2 N$ **5.** $4 \log_2 M - 3 \log_2 N$ **7.** $\frac{1}{2} \log_2 M - \frac{3}{2} \log_2 N$ **9.** 1.90 **11.** 0.15 **13.** 0.90 **15.** 0.35 **17.** -0.95 **19.** -0.22 **21.** $\log_4 p^5q$ **23.** $\log_3 \frac{A^4}{\sqrt{B}}$ **25.** $\log_2 8MN$ **27.** $\log_5 \frac{5}{x^3}$ **29.** 2 **31.** $\frac{3}{2}$ **33.** $\{45\}$ **35.** $\{1\}$ **37.** $\{6\}$ **39.** $\{\pm 5\}$ **41. a.** 6 **b.** $\frac{1}{4}$ **c.** 1 **45.** $\{3\}$ **47.** $\{2\}$ **49.** $\{\sqrt{85}\}$ **51.** $\{2\}$

Mixed Review Exercises, page 477 **1.** $\left\{\frac{\sqrt{2}}{2}\right\}$ **2.** $\{-3, 1, 2\}$ **3.** $\{5\}$ **4.** $\{4\}$ **5.** $\left\{-\frac{3}{2}\right\}$ **6.** $\{7\}$ **7.** $\left\{-\frac{1}{2}\right\}$ **8.** $\{4\}$ **9.** $\{2 \pm \sqrt{5}\}$ **10.** 3 **11.** 2 **12.** 1 **13.** 1

Self-Test 2, page 477 **1. a.** $3^4 = 81$ **b.** $6^3 = 216$ **2. a.** $\log_5 625 = 4$ **b.** $\log_{25} 125 = \frac{3}{2}$ **3. a.** 3 **b.** 12 **4.** $\{3\}$ **5.** $\frac{5}{3} \log_2 M + 2 \log_2 N$ **6.** -1.40 **7.** $\{3\}$

Written Exercises, pages 481–482 **1.** 1.79 **3.** 0.00792 **5.** 575 **7.** 33.7 **9.** 7.13 **11.** 692 **13.** 0.0158 **15. a.** $\frac{\log 30}{\log 3}$ **b.** 3.10

17. a. $\frac{\log 56}{\log 5.6}$ **b.** 2.34 **19. a.** $-\frac{\log 5}{\log 30}$ **b.** −0.473 **21. a.** $\frac{\log 60}{2 \log 3.5}$ **b.** 1.63 **23.** $\left\{\frac{7}{4}\right\}$ **25.** $\left\{\frac{5}{6}\right\}$ **27.** {6740} **29.** {21.6} **31.** {2.19} **33.** {1.03} **35.** 3.17 **37.** 3.36 **39.** {0.631, 1.46} **41. a.** 2; $\frac{1}{2}$ **b.** 3; $\frac{1}{3}$ **c.** They are reciprocals. $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$

Problems, pages 486–488 **1. a.** \$1120 **b.** \$1254.40 **c.** \$1404.93 **d.** \$3105.85 **3. a.** \$1125.51 **b.** \$1266.77 **c.** \$1425.76 **d.** \$3262.04 **5. a.** \$10,000 **b.** \$8000 **c.** \$6400 **d.** \$1342.18 **7. a.** $4N_0$ **b.** $32N_0$ **c.** $(2^{w/3})N_0$ **9. a.** 25 kg **b.** 6.25 kg **c.** $100\left(\frac{1}{2}\right)^{y/6000}$ kg **11.** about 9 years **13.** 12.6% **15.** 0.997; 9.09×10^{-13} **17.** 11.4% **19.** 19.7% **21. a.** \$1,065,552.45 **b.** \$1,066,086.39 **c.** \$1,066,091.81

Mixed Review Exercises, page 488
1. $\left\{0, \frac{1}{2}, 1\right\}$ **2.** {1, 2} **3.** $\{\sqrt{2}, 2, 4\}$ **4.** {0, 6} **5.** $\left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}$ **6.** $\{1, \sqrt{2}, 2\}$ **7.** 1 **8.** 2 **9.** $\frac{1}{6}$ **10.** 1 **11.** $\frac{5}{12}$ **12.** 3

Written Exercises, pages 490–491 **1.** $e^{2.08} = 8$ **3.** $e^{-1.10} = \frac{1}{3}$ **5.** $\ln 20.1 = 3$ **7.** $\ln 1.65 = \frac{1}{2}$ **9.** 2 **11.** −3 **13.** 0 **15.** 5 **17.** ln 12 **19.** $\ln \frac{9}{5}$ **21.** $\ln 10e^3$ **23.** $\{e^3\}$ **25.** $\{4 + e^{-1}\}$ **27.** $\{\pm e^{9/2}\}$ **29.** {ln 2} **31.** {ln 5} **33.** {2 + ln 2} **35.** {ln 9} **37.** $\left\{\frac{3}{5} \ln 10\right\}$ **39.** {e} **41.** {2} **43.** {ln 3, ln 4} **45.** $D = \{x: x \neq 0\}$; $R = \{\text{reals}\}$ **47.** $D = \{x: x > 5\}$; $R = \{\text{reals}\}$

Calculator Key-In, page 492 **1. a.** \$1061.84 **b.** \$1127.50 **c.** \$1822.12 **3. a.** 1; 0.99; 0.96; 0.91; 0.85; 0.78; 0.70; 0.61; 0.53; 0.44; 0.37 **5.** 2.718056; *e*

Self-Test 3, page 493 **1.** {650} **2.** {5.81} **3.** {1.07} **4.** \$3277.23 **5.** $\ln\left(\frac{5}{e^2}\right)^{1/3}$ $\left(\text{or } \frac{\ln 5 - 2}{3}\right)$

Application, pages 493–494 **1.** 6600 **3. a.** 95.3% **b.** 93.0%

Chapter Review, pages 496–497 **1.** c **3.** d **5.** b **7.** a **9.** b **11.** a **13.** d **15.** b

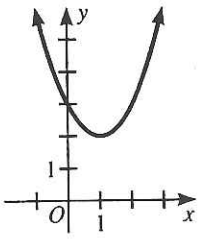
Mixed Problem Solving, page 498 **1.** 10 mL **3.** CA:45; NY:34; NC:11 **5.** −9 **7.** \$1500 **9.** $A = \frac{C^2}{4\pi}$ **11.** Larger plant: 8.4 h; smaller: 12.4 h **13.** 4 h

Preparing for College Entrance Exams, page 499 **1.** C **3.** E **5.** A **7.** E

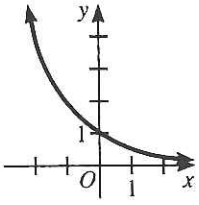
Chapter 11 Sequences and Series

Written Exercises, pages 504–506 **1.** A; 8, 5 **3.** G; 625, 3125 **5.** A; 30, 38 **7.** N; $\frac{1}{25}, \frac{1}{36}$ **9.** G; $4^{9/2}, 4^{11/2}$ **11.** 7, 11, 15, 19; A **13.** 1, 3, 9, 27; G **15.** $-\frac{1}{4}, \frac{1}{2}, -1, 2$; G **17.** log 2, log 3, log 4, log 5; N **19. a.** A **b.** G **21.** 32, 44 **23.** 20, 18 **25.** 63; 127 **27.** 21, 34 **29.** 48, 71 **31. a.** 15, 21 **b.** 55 **33. a.** 9 **b.** 35 **35. a.** 16, 31; No **b.** 57

Mixed Review Exercises, page 506
1.



2.



7. (2, −3) **8.** $\left(-\frac{3}{2}, \frac{7}{2}\right)$, (1, 1) **9.** (2, 1), (−2, 1), (2, −1), (−2, −1)

Written Exercises, page 509 **1.** $t_n = 8n + 16$ **3.** $t_n = 4 - 7n$ **5.** $t_n = 4n + 3$ **7.** 104 **9.** 52 **11.** −902 **13.** 87.5 **15.** 8 **17.** −61 **19.** 2 **21.** $\frac{3}{2}$ **23. a.** −7, 13 **b.** −12, 3, 18 **c.** −15, −3, 9, 21 **25. a.** 19, 27 **b.** 17, 23, 29 **c.** 15.8, 20.6, 25.4, 30.2 **27.** 101 **29.** 300 **31.** 16 **33.** $\frac{a+b}{2} - a = b - \frac{a+b}{2} = \frac{b-a}{2}$