

$$\begin{aligned} 3 \quad \frac{P(x)}{x-2} &= \frac{(x-2)(x^2+2x+3)+7}{x-2} \\ &= x^2+2x+3+\frac{7}{x-2} \end{aligned}$$

∴ quotient is x^2+2x+3 ,
remainder is 7

$$\begin{aligned} 4 \quad \frac{f(x)}{x^2+x-2} &= \frac{(x-1)(x+2)(x^2-3x+5)+15-10x}{(x-1)(x+2)} \\ &= x^2-3x+5+\frac{15-10x}{(x-1)(x+2)} \end{aligned}$$

∴ quotient is x^2-3x+5 , remainder is $15-10x$

EXERCISE 7D.1

1 a $2x^2 - 5x - 12$ has zeros
 $x = \frac{5 \pm \sqrt{25 - 4(2)(-12)}}{4}$

$$= \frac{5 \pm \sqrt{121}}{4}$$

$$= \frac{5 \pm 11}{4}$$

$$= 4, -\frac{6}{4}$$

$$\therefore \text{zeros are } 4, -\frac{3}{2}$$

c $z^2 - 6z + 6$ has zeros
 $z = \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2}$

$$= \frac{6 \pm \sqrt{12}}{2}$$

$$= 3 \pm \sqrt{3}$$

$$\therefore \text{zeros are } 3 \pm \sqrt{3}$$

e $z^3 + 2z$
 $= z(z^2 + 2)$
 $= z(z^2 - 2i^2)$
 $= z(z + i\sqrt{2})(z - i\sqrt{2})$
 $\therefore \text{zeros are } 0, \pm i\sqrt{2}$

2 a $5x^2 = 3x + 2$

$$\therefore 5x^2 - 3x - 2 = 0$$
 $\therefore (5x+2)(x-1) = 0$

$$\therefore \text{roots are } 1, -\frac{2}{5}$$

c $-2z(z^2 - 2z + 2) = 0$
 $z = 0 \text{ or } \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2}$

$$= 0 \text{ or } \frac{2 \pm \sqrt{-4}}{2}$$

$$= 0 \text{ or } 1 \pm i$$

$$\therefore \text{roots are } 0, 1 \pm i$$

e $z^3 + 5z = 0$
 $z(z^2 + 5) = 0$
 $z(z^2 - 5i^2) = 0$
 $z(z + i\sqrt{5})(z - i\sqrt{5}) = 0$
 $\therefore \text{roots are } 0, \pm i\sqrt{5}$

b $x^2 + 6x + 10$ has zeros
 $x = \frac{-6 \pm \sqrt{36 - 4(1)(10)}}{2}$
 $= \frac{-6 \pm \sqrt{-4}}{2}$
 $= -3 \pm i$

d $x^3 - 4x$
 $= x(x^2 - 4)$
 $= x(x+2)(x-2)$

$$\therefore \text{zeros are } 0, \pm 2$$

f $z^4 + 4z^2 - 5$
 $= (z^2 + 5)(z^2 - 1)$
 $= (z^2 - 5i^2)(z^2 - 1)$
 $= (z + i\sqrt{5})(z - i\sqrt{5})(z + 1)(z - 1)$

$$\therefore \text{zeros are } \pm i\sqrt{5}, \pm 1$$

b $(2x+1)(x^2+3)=0$
 $\therefore (2x+1)(x^2-3i^2)=0$
 $\therefore (2x+1)(x+i\sqrt{3})(x-i\sqrt{3})=0$

$$\therefore \text{roots are } -\frac{1}{2}, \pm i\sqrt{3}$$

d $x^3 = 5x$
 $\therefore x^3 - 5x = 0$
 $x(x^2 - 5) = 0$
 $x(x + \sqrt{5})(x - \sqrt{5}) = 0$

$$\therefore \text{roots are } 0, \pm \sqrt{5}$$

f $z^4 = 3z^2 + 10$
 $\therefore z^4 - 3z^2 - 10 = 0$
 $(z^2 - 5)(z^2 + 2) = 0$
 $(z^2 - 5)(z^2 - 2i^2) = 0$
 $(z + \sqrt{5})(z - \sqrt{5})(z + i\sqrt{2})(z - i\sqrt{2}) = 0$

$$\therefore \text{roots are } \pm \sqrt{5}, \pm i\sqrt{2}$$

3 a $2x^2 - 7x - 15$
 $= (2x+3)(x-5)$

c $x^3 + 2x^2 - 4x$
 $= x(x^2 + 2x - 4)$
 $x^2 + 2x - 4$ is zero when
 $x = \frac{-2 \pm \sqrt{4+16}}{2}$

$$= -1 \pm \sqrt{5}$$

$$\therefore x^3 + 2x^2 - 4x$$

 $= x(x+1+\sqrt{5})(x+1-\sqrt{5})$

$$e z^4 - 6z^2 + 5$$

$$= (z^2 - 1)(z^2 - 5)$$

$$= (z+1)(z-1)(z+\sqrt{5})(z-\sqrt{5})$$

4 $P(x) = a(x - \alpha)(x - \beta)(x - \gamma)$
 $\therefore P(\alpha) = a \times 0 \times (\alpha - \beta)(\alpha - \gamma) = 0$
and $P(\beta) = a(\beta - \alpha) \times 0 \times (\beta - \gamma) = 0$
and $P(\gamma) = a(\gamma - \alpha)(\gamma - \beta) \times 0 = 0$

5 a The zeros ± 2
have sum = 0 and product = -4
∴ come from quadratic factor $z^2 - 4$
and zero 3 comes from $(z-3)$
∴ $P(z) = a(z^2 - 4)(z-3)$, $a \neq 0$

c The zeros $-1 \pm i$
have sum = -2 and product = 2
∴ come from quadratic factor $z^2 + 2z + 2$
and zero -1 comes from $(z+1)$
∴ $P(z) = a(z-3)(z^2 + 2z + 2)$, $a \neq 0$

6 a For zeros of ± 1 , sum = 0 and product = -1
For zeros of $\pm \sqrt{2}$, sum = 0 and product = -2
∴ $P(z) = a(z^2 - 1)(z^2 - 2)$, $a \neq 0$

b For zeros of $\pm i\sqrt{3}$, sum = 0 and product = 3
zeros of 2, -1 come from $(z-2)(z+1)$
∴ $P(z) = a(z-2)(z+1)(z^2 + 3)$, $a \neq 0$

c For zeros of $\pm \sqrt{3}$, sum = 0 and product = -3
For zeros of $1 \pm i$, sum = 2 and product = 2
∴ $P(z) = a(z^2 - 3)(z^2 - 2z + 2)$, $a \neq 0$

d For zeros of $2 \pm \sqrt{5}$, sum = 4 and product = -1
For zeros of $-2 \pm 3i$, sum = -4 and product = 13
∴ $P(z) = a(z^2 - 4z - 1)(z^2 + 4z + 13)$, $a \neq 0$

EXERCISE 7D.2

1 a $2x^2 + 4x + 5 = ax^2 + [2b-6]x + c$
Equating coefficients gives
 $a = 2$, $2b-6 = 4$, and $c = 5$

$$\therefore 2b = 10$$

$$\therefore b = 5$$

$$\therefore a = 2, b = 5, c = 5$$

b $2x^3 - x^2 + 6 = (x-1)^2(2x+a) + bx + c$
 $= (x^2 - 2x + 1)(2x+a) + bx + c$
 $= 2x^3 + [a-4]x^2 + [2-2a]x + a + bx + c$
 $= 2x^3 + [a-4]x^2 + [2-2a+b]x + [a+c]$

Equating coefficients gives $a-4=-1$ $2-2a+b=0$ $a+c=6$
 $\therefore a=3$ $\therefore b=2a-2$ $\therefore c=6-a$
 $\therefore b=4$ $\therefore c=3$

2 a $z^4 + 4 = (z^2 + az + 2)(z^2 + bz + 2)$
 $= z^4 + [a+b]z^3 + [4+ab]z^2 + [2a+2b]z + 4$

Equating coefficients gives: $a+b=0$ $4+ab=0$
 $\therefore a=-b$ $\therefore ab=-4$

By inspection $a=2$ and $b=-2$
or $a=-2$ and $b=2$

$$\begin{array}{r} & 1 & a & 2 \\ \times & 1 & b & 2 \\ \hline & 2 & 2a & 4 \\ & b & ab & 2b \\ \hline 1 & a & 2 \\ \hline 1 & a+b & 4+ab & 2a+2b & 4 \end{array}$$

b $2z^4 + 5z^3 + 4z^2 + 7z + 6$
 $= (z^2 + az + 2)(2z^2 + bz + 3)$
 $= 2z^4 + [2a+b]z^3 + [ab+7]z^2 + [3a+2b]z + 6$

Equating coefficients gives: $2a+b=5$ (1)
 $3a+2b=7$ (2)
 $ab+7=4$ (3)
 $\therefore 4a+2b=10$ { (1) $\times 2$ }
and $3a+2b=7$

and solving these two equations gives $a=3$, $b=-1$
which checks with (3) as $ab+7=-3+7=4$ ✓

$$\begin{array}{r} & 1 & a & 2 \\ \times & 2 & b & 3 \\ \hline & 3 & 3a & 6 \\ & b & ab & 2b \\ \hline 2 & 2a & 4 \\ \hline 2 & 2a+b & ab+7 & 3a+2b & 6 \end{array}$$

3 Consider
 $z^4 + 64 = (z^2 + az + 8)(z^2 + bz + 8)$
 $= z^4 + [a+b]z^3 + [ab+16]z^2 + [8a+8b]z + 64$

Equating coefficients gives:
 $a+b=0$ and $ab+16=0$
 $\therefore a=-b$ $\therefore ab=-16$
 \therefore by inspection $a=4$ and $b=-4$
or $a=-4$ and $b=4$
 $\therefore z^4 + 64$ can be factorised into $(z^2 + 4z + 8)(z^2 - 4z + 8)$

$$\begin{array}{r} & 1 & a & 8 \\ \times & 1 & b & 8 \\ \hline & 8 & 8a & 64 \\ & b & ab & 8b \\ \hline 1 & a & 8 \\ \hline 1 & a+b & ab+16 & 8a+8b & 64 \end{array}$$

Now consider
 $z^4 + 64 = (z^2 + az + 16)(z^2 + bz + 4)$
 $= z^4 + [a+b]z^3 + [ab+20]z^2 + [4a+16b]z + 64$

Equating coefficients gives:
 $a+b=0$ (1) and $ab+20=0$
 $4a+16b=0$ (2) $\therefore ab=-20$ (3)

Solution to (1), (2) is $a=b=0$

But this does not satisfy (3)

\therefore no values of a and b exist which obey the original assumption

\therefore cannot be factorised in this way.

$$\begin{array}{r} & 1 & a & 16 \\ \times & 1 & b & 4 \\ \hline & 4 & 4a & 64 \\ & b & ab & 16b \\ \hline 1 & a & 16 \\ \hline 1 & a+b & ab+20 & 4a+16b & 64 \end{array}$$

4 Consider

$$\begin{aligned} x^4 - 4x^2 + 8x - 4 &= (x^2 + ax + 2)(x^2 + bx - 2) \\ &= x^4 + [a+b]x^3 + [ab]x^2 + [2b-2a]x - 4 \end{aligned}$$

Equating coefficients gives:

$$\begin{aligned} a+b &= 0 \quad \text{and} \quad ab = -4 \quad \text{and} \quad -2a+2b = 8 \\ \therefore 2a+2b &= 0 \quad \dots (1) \\ -2a+2b &= 8 \quad \dots (2) \end{aligned}$$

Adding (1) and (2) gives $4b = 8$ $\therefore b = 2$ and hence $a = -2$, which checks with $ab = -4$ ✓

$$\therefore P(x) = (x^2 - 2x + 2)(x^2 + 2x - 2)$$

Now if $x^4 + 8x = 4x^2 + 4$

$$\text{then } x^4 - 4x^2 + 8x - 4 = 0$$

$$\therefore (x^2 - 2x + 2)(x^2 + 2x - 2) = 0$$

$$\therefore x^2 - 2x + 2 = 0 \quad \text{or} \quad x^2 + 2x - 2 = 0$$

$$\therefore x = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i \quad \text{or} \quad x = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$$

$$\therefore x = 1 \pm i, -1 \pm \sqrt{3}$$

5 a $P(z) = 2z^3 - z^2 + az - 3$

$$\begin{aligned} &= (2z-3)(z^2 + bz + 1) \quad \text{for some value } b \\ &= 2z^3 + [2b-3]z^2 + [2-3b]z - 3 \end{aligned}$$

Equating coefficients gives:

$$\begin{aligned} 2b-3 &= -1 \quad \text{and} \quad 2-3b = a \\ 2b &= 2 \quad \therefore a = 2-3b \\ b &= 1 \quad \therefore a = -1 \end{aligned}$$

$$\therefore P(z) = (2z-3) \underbrace{(z^2 + z + 1)}$$

this quadratic has zeros $z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$$\therefore a = -1 \quad \text{and zeros are } \frac{3}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

b $P(z) = 3z^3 - z^2 + [a+1]z + a$

$$\begin{aligned} &= (3z+2)(z^2 + bz + c) \\ &= 3z^3 + [2+3b]z^2 + [2b+3c]z + 2c \end{aligned}$$

Equating coefficients gives:

$$\therefore 2+3b=-1, 2b+3c=a+1 \quad \text{and} \quad 2c=a$$

Now as $2+3b=-1$

$$\therefore 3b=-3$$

$$\therefore b=-1$$

Substituting $b=-1$ and $a=2c$ into $2b+3c=a+1$ gives $2(-1)+3c=2c+1$

$$\therefore -2+3c=2c+1$$

$$c=3$$

and so $a=6$

$$\therefore P(z) = (3z+2) \underbrace{(z^2 - z + 3)}$$

this quadratic has zeros $\frac{1 \pm \sqrt{1-4(3)(1)}}{2} = \frac{1 \pm i\sqrt{11}}{2}$

So, $a=6$ and the zeros are $-\frac{2}{3}, \frac{1}{2} \pm i\frac{\sqrt{11}}{2}$.

$$\begin{aligned} 6 \quad a \quad P(x) &= 2x^4 + ax^3 + bx^2 - 12x - 8 \\ &= (2x+1)(x-2)(x^2 + cx + 4) \\ &= (2x^2 - 3x - 2)(x^2 + cx + 4) \end{aligned}$$

Equating coefficients: $2c - 3 = a$,
 $6 - 3c = b$ and $-2c - 12 = -12$
The last equation has solution $c = 0$, and consequently,

$$a = -3 \text{ and } b = 6$$

$$\therefore P(x) = (2x+1)(x-2)(x^2 + 4) = (2x+1)(x-2)(x+2i)(x-2i)$$

∴ zeros are $-\frac{1}{2}, 2$ and $\pm 2i$ and $a = -3, b = 6$.

$$\begin{aligned} b \quad P(x) &= 2x^4 + ax^3 + bx^2 + ax + 3 \\ &= (x+3)(2x-1)(x^2 + cx - 1) \\ &= (2x^2 + 5x - 3)(x^2 + cx - 1) \end{aligned}$$

Equating coefficients: $a = 2c + 5$,
 $b = 5c - 5$, $a = -5 - 3c$

$$\therefore 2c + 5 = -5 - 3c \quad \{\text{equating } a\text{'s}\}$$

$$\therefore 5c = -10$$

$$\therefore c = -2 \text{ and so, } a = 1, b = -15$$

$$\therefore P(x) = (x+3)(2x-1)(x^2 - 2x - 1)$$

this quadratic has zeros $\frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

∴ zeros are $-3, \frac{1}{2}, 1 \pm \sqrt{2}$ and $a = 1, b = -15$

$$\begin{aligned} 7 \quad a \quad x^3 + 3x^2 - 9x + c &= (x+a)^2(x+b) \\ &= (x^2 + 2ax + a^2)(x+b) \\ &= x^3 + [b + 2a]x^2 + [a^2 + 2ab]x + a^2b \end{aligned}$$

Equating coefficients gives

$$2a + b = 3, a^2 + 2ab = -9 \text{ and } c = a^2b$$

Substituting $b = 3 - 2a$ into the second equation gives:

$$a^2 + 2a(3 - 2a) = -9$$

$$\therefore a^2 + 6a - 4a^2 = -9$$

$$\therefore -3a^2 + 6a = -9$$

$$\therefore 3a^2 - 6a - 9 = 0$$

$$\therefore a^2 - 2a - 3 = 0$$

$$\therefore (a-3)(a+1) = 0$$

$$b \quad 3x^3 + 4x^2 - x + m$$

$$= (x+a)^2(3x+b)$$

$$= (x^2 + 2ax + a^2)(3x+b)$$

$$= x^3 + [6a+b]x^2 + [3a^2 + 2ab]x + a^2b$$

Equating coefficients gives

$$6a + b = 4, 3a^2 + 2ab = -1 \text{ and } a^2b = m$$

Substituting $b = 4 - 6a$ into the second equation gives:

$$\begin{aligned} &3a^2 + 2a(4 - 6a) = -1 \\ &\therefore 3a^2 + 8a - 12a^2 = -1 \\ &\therefore 9a^2 - 8a - 1 = 0 \\ &\therefore (9a+1)(a-1) = 0 \\ &\therefore a = -\frac{1}{9} \text{ or } a = 1 \end{aligned}$$

When $a = 1, b = -2$ and $m = -2$. So, $P(x) = (x+1)^2(3x-2)$

When $a = -\frac{1}{9}, b = \frac{14}{3}$ and $m = \frac{14}{243}$. So, $P(x) = (x - \frac{1}{9})^2(3x + \frac{14}{3})$

$$\begin{array}{r} & 2 & -3 & -2 \\ \times & 1 & c & 4 \\ \hline & 8 & -12 & -8 \\ & 2c & -3c & -2c \\ \hline 2 & -3 & -2 \\ \hline 2 & 2c-3 & 6-3c & -2c-12 & -8 \end{array}$$

$$\begin{array}{r} & 2 & 5 & -3 \\ \times & 1 & c & -1 \\ \hline & -2 & -5 & 3 \\ & 2c & 5c & -3c \\ \hline 2 & 5 & -3 \\ \hline 2 & 2c+5 & 5c-5 & -5-3c & 3 \end{array}$$

EXERCISE 7D.3

1 a If $P(2) = 7$, then $P(x) = (x-2)Q(x) + 7$ and $P(x)$ divided by $(x-2)$ leaves a remainder of 7.

b If $P(x) = (x+3)Q(x) - 8$, then $P(-3) = -8$ and $P(x)$ divided by $(x+3)$ leaves a remainder of -8.

c If $P(x)$ when divided by $(x-5)$ has a remainder of 11, then $P(5) = 11$ and $P(x) = (x-5)Q(x) + 11$.

$$2 \quad a \quad P(x) = x^3 + 2x^2 - 7x + 5$$

$$\begin{aligned} &\therefore R = P(1) \quad \{\text{Remainder theorem}\} \\ &= 1^3 + 2(1)^2 - 7 + 5 \\ &= 1 \end{aligned}$$

$$b \quad P(x) = x^4 - 2x^2 + 3x - 1$$

$$\begin{aligned} &\therefore R = P(-2) \quad \{\text{Remainder theorem}\} \\ &= (-2)^4 - 2(-2)^2 + 3(-2) - 1 \\ &= 16 - 8 - 6 - 1 \\ &= 1 \end{aligned}$$

$$3 \quad a \quad P(x) = x^3 - 2x + a$$

$$\begin{aligned} &\text{Now } P(2) = 7 \quad \{\text{Remainder theorem}\} \\ &\therefore 2^3 - 2(2) + a = 7 \\ &4 + a = 7 \\ &\therefore a = 3 \end{aligned}$$

$$b \quad P(x) = 2x^3 + x^2 + ax - 5$$

$$\begin{aligned} &\text{Now } P(-1) = -8 \\ &\therefore 2(-1)^3 + (-1)^2 + a(-1) - 5 = -8 \\ &-2 + 1 - a - 5 = -8 \\ &-a - 6 = -8 \\ &-a = -2 \\ &\therefore a = 2 \end{aligned}$$

$$4 \quad P(x) = x^3 + 2x^2 + ax + b$$

Now $P(1) = 4$ and $P(-2) = 16$ {Remainder theorem}

If $P(1) = 4$ then $1 + 2 + a + b = 4$ and so $a + b = 1$ (1)

If $P(-2) = 16$ then $(-2)^3 + 2(-2)^2 + a(-2) + b = 16$

$$\therefore -8 + 8 - 2a + b = 16$$

$$\therefore -2a + b = 16 \quad \dots\dots (2)$$

Solving (1) and (2) $\begin{aligned} &-a - b = -1 \\ &-2a + b = 16 \end{aligned}$

$$\begin{aligned} &\therefore -3a = 15 \quad \{\text{adding}\} \\ &\therefore a = -5 \quad \text{and so } b = 6 \\ &\therefore a = -5 \quad \text{and } b = 6 \end{aligned}$$

$$5 \quad P(x) = 2x^n + ax^2 - 6$$

By the Remainder theorem, $P(1) = -7 \therefore 2(1)^n + a(1)^2 - 6 = -7$

$$\therefore 2 + a - 6 = -7$$

$$\therefore a = -3$$

So, $P(x) = 2x^n - 3x^2 - 6$

and since $P(-3) = 129, \therefore 2(-3)^n - 3(-3)^2 - 6 = 129$

$$2(-3)^n - 27 - 6 = 129$$

$$2(-3)^n = 162$$

$$(-3)^n = 81$$

$$\therefore n = 4$$

$$\therefore a = -3 \text{ and } n = 4$$

$$P(z) = Q(z)(z^2 - 3z + 2) + (4z - 7) = Q(z)(z-2)(z-1) + (4z - 7)$$

Remainder is $P(1)$ {Remainder theorem}

$$\therefore R = Q(1) \times (1-2) \times 0 + (4-7)$$

$$= -3$$

Remainder is $P(2)$ {Remainder theorem}

$$\therefore R = Q(2) \times 0 \times (2-1) + [4(2)-7]$$

$$= 0 + 1$$

$$= 1$$

7 Suppose $P(z)$ is divided by $(z - 3)(z + 1)$
 $\therefore P(z) = Q(z) \times (z - 3)(z + 1) + (Az + B)$
 \uparrow
the remainder must be of this form
Now $P(-1) = -8 \quad \therefore Q(-1) \times 0 + (-A + B) = -8$
 $\therefore -A + B = -8 \quad \dots\dots (1)$

and $P(3) = 4 \quad \therefore Q(3) \times 0 + (3A + B) = 4$
 $\therefore 3A + B = 4 \quad \dots\dots (2)$

Solving (1) and (2)
 $\begin{array}{r} -A + B = -8 \\ -3A - B = -4 \\ \hline -4A = -12 \\ \therefore A = 3 \text{ and so } B = -5 \\ \therefore R(z) = 3z - 5 \end{array}$

8 Suppose $P(x)$ is divided by $(x - a)(x - b)$ and has remainder $Ex + F$
hence $P(x) = Q(x) \times (x - a)(x - b) + Ex + F$
Now $P(a) = Ea + F \quad \dots\dots (1)$ and $P(b) = Eb + F \quad \dots\dots (2)$

Subtracting (2) and (1), $P(b) - P(a) = Eb - Ea = E(b - a)$
 $\therefore E = \frac{P(b) - P(a)}{b - a}$
from (1) $F = P(a) - Ea = P(a) - \left(\frac{P(b) - P(a)}{b - a}\right)a$
Now $R(x) = Ex + F$
 $\therefore R(x) = \left(\frac{P(b) - P(a)}{b - a}\right)x + P(a) - \left(\frac{P(b) - P(a)}{b - a}\right)a$
 $\therefore R(x) = \left(\frac{P(b) - P(a)}{b - a}\right)(x - a) + P(a)$

EXERCISE 7D.4

1 a $P(x) = 2x^3 + x^2 + kx - 4$
if $x + 2$ is a factor then $P(-2) = 0$
 $\therefore -2k - 16 = 0$
 $\therefore k = -8$

$\therefore P(x) = 2x^3 + x^2 - 8x - 4$
 $= (x + 2)(2x^2 - 3x - 2) \quad \{\text{as when } k = -8, k + 6 = -2\}$
 $\therefore P(x) = (x + 2)(2x + 1)(x - 2) \text{ and } k = -8$

$$\begin{array}{r} -2 \\ \hline 2 & 1 & k & -4 \\ 0 & -4 & 6 & -2k - 12 \\ \hline 2 & -3 & k + 6 & -2k - 16 \end{array}$$

b $P(x) = x^4 - 3x^3 - kx^2 + 6x$
if $x - 3$ is a factor then $P(3) = 0$
 $\therefore 18 - 9k = 0$

$\therefore 9k = 18$
 $\therefore k = 2$
 $\therefore P(x) = x^4 - 3x^3 - 2x^2 + 6x$
 $P(x) = (x - 3)(x^3 - 2x) \quad \{\text{as when } k = 2, -k = -2 \text{ and } 6 - 3k = 0\}$
 $= x(x - 3)(x^2 - 2)$
 $= x(x - 3)(x + \sqrt{2})(x - \sqrt{2}) \text{ and } k = 2$

$$\begin{array}{r} 3 \\ \hline 1 & -3 & -k & 6 & 0 \\ 0 & 3 & 0 & -3k & 18 - 9k \\ \hline 1 & 0 & -k & 6 - 3k & 18 - 9k \end{array}$$

2 $P(x) = 2x^3 + ax^2 + bx + 5$
if $x - 1$ is a factor, $P(1) = 0$
 $\therefore 2(1)^3 + a(1)^2 + b(1) + 5 = 0$
 $2 + a + b + 5 = 0$
 $\therefore a + b = -7 \quad \dots\dots (1)$

Adding (1) and (2) gives: $6a = 42$
 $\therefore a = 7 \text{ and } b = -14$

3 a $P(z) = z^3 - z^2 + [k - 5]z + [k^2 - 7]$
if 3 is a zero, $R = P(3) = 0$
 $\therefore k^2 + 3k - 4 = 0$
 $(k + 4)(k - 1) = 0$
 $\therefore k = -4 \text{ or } k = 1$

$$\begin{array}{r} 3 \\ \hline 1 & -1 & k - 5 & k^2 - 7 \\ 0 & 3 & 6 & 3k + 3 \\ \hline 1 & 2 & k + 1 & k^2 + 3k - 4 \end{array}$$

if $k = 1$, $P(z) = (z - 3)(z^2 + 2z + 2)$
the quadratic has zeros: $\frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$
 \therefore zeros are 3, $-1 \pm i$

b $P(z) = z^3 + mz^2 + (3m - 2)z - 10m - 4$
if $z - 2$ is a factor, $P(2) = 0$
since $* = 0$ R_1 is always 0
 $\therefore z - 2$ is always a factor
now for $(z - 2)^2$ to be a factor
 $7m + 10 = 0 \quad \{R_2 \text{ is also } 0\} \quad \therefore m = -\frac{10}{7}$

$$\begin{array}{r} 2 \\ \hline 1 & m & 3m - 2 & -10m - 4 \\ 0 & 2 & 2m + 4 & 10m + 4 \\ \hline 1 & m + 2 & 5m + 2 & 0 \dots (*) \\ 0 & 2 & 2m + 8 & \\ \hline 1 & m + 4 & 7m + 10 & \end{array}$$

4 a i $P(x) = x^3 - a^3$
 $P(a) = a^3 - a^3$
 $= 0$
 $\therefore x - a$ is a linear factor of $P(x)$ for all a

$$\begin{array}{r} a \\ \hline 1 & 0 & 0 & -a^3 \\ 0 & a & a^2 & a^3 \\ \hline 1 & a & a^2 & 0 \end{array}$$

$\therefore P(x) = (x - a)(x^2 + ax + a^2)$

ii $-a \quad \begin{array}{r} 1 & 0 & 0 & a^3 \\ 0 & -a & a^2 & -a^3 \\ \hline 1 & -a & a^2 & 0 \end{array}$
 $\therefore P(x) = (x + a)(x^2 - ax + a^2)$

5 a Consider $P(x) = x^n + 1$
if $x + 1$ is a factor then $P(-1) = 0$
 $\therefore (-1)^n + 1 = 0$
 $\therefore (-1)^n = -1$
which is only true if n is odd
 $\therefore x + 1$ is a factor of $x^n + 1 \Leftrightarrow n$ is odd.

b $P(x) = x^3 - 3ax - 9$ and if $x - 1 - a$ is a factor then $P(1 + a) = 0$
 $1 + a \quad \begin{array}{r} 1 & 0 & -3a & -9 \\ 0 & 1 + a & 1 + 2a + a^2 & a^3 + 1 \\ \hline 1 & 1 + a & a^2 - a + 1 & a^3 - 8 \end{array}$
 $\therefore a^3 - 8 = 0$
 $\therefore a = 2 \quad \{\text{the only real soln.}\}$

EXERCISE 7E.1

- 1** **a** A single factor such as $(x - \alpha)$ indicates that the graph *cuts* the x -axis at α .
b A squared factor such as $(x - \alpha)^2$ indicates that the graph *touches* the x -axis at α .
c A cubed factor such as $(x - \alpha)^3$ indicates that the graph *cuts* the x -axis at α , and at α the graph changes shape.

- 2** **a** The x -intercepts are: $-1, 2$ and 3 $\therefore y = a(x + 1)(x - 2)(x - 3)$, $a \neq 0$
As the curve passes through $(0, 12)$,
 $12 = a(1)(-2)(-3) \quad \therefore a = 2$
 $\therefore y = 2(x + 1)(x - 2)(x - 3)$
- b** The x -intercepts are: $-3, -\frac{1}{2}$ and $\frac{1}{2}$ $\therefore y = a(x + 3)(2x + 1)(2x - 1)$, $a \neq 0$
As the curve passes through $(0, 6)$,
 $6 = a(3)(1)(-1) \quad \therefore a = -2$
 $\therefore y = -2(x + 3)(2x + 1)(2x - 1)$
- c** The x -intercepts are: $-4, -4$ and 3 $\therefore y = a(x + 4)^2(x - 3)$, $a \neq 0$
As the curve passes through $(0, -12)$,
 $-12 = a(4)^2(-3) \quad \therefore a = \frac{1}{4}$
 $\therefore y = \frac{1}{4}(x + 4)^2(x - 3)$
- d** The x -intercepts are: $-5, -2$ and 5 $\therefore y = a(x + 5)(x + 2)(x - 5)$, $a \neq 0$
As the curve passes through $(0, -5)$,
 $-5 = a(5)(2)(-5) \quad \therefore a = \frac{1}{10}$
 $\therefore y = \frac{1}{10}(x + 5)(x + 2)(x - 5)$
- e** The x -intercepts are: $-4, 3$ and 3 $\therefore y = a(x + 4)(x - 3)^2$, $a \neq 0$
As the curve passes through $(0, 9)$,
 $9 = a(4)(-3)^2 \quad \therefore a = \frac{1}{4}$
 $\therefore y = \frac{1}{4}(x + 4)(x - 3)^2$
- f** The x -intercepts are: $-3, -2$ and $-\frac{1}{2}$ $\therefore y = a(x + 3)(x + 2)(2x + 1)$, $a \neq 0$
As the curve passes through $(0, -12)$,
 $-12 = a(3)(2)(1) \quad \therefore a = -2$
 $\therefore y = -2(x + 3)(x + 2)(2x + 1)$

3 **a** $P(x) = a(x - 3)(x - 1)(x + 2)$

Since $P(x)$ passes through $(2, -4)$

$$-4 = a(-1)(4)$$

$$\therefore -4 = -4a$$

$$\therefore a = 1$$

$$\therefore P(x) = (x - 3)(x - 1)(x + 2)$$

c $P(x) = a(x - 1)^2(x + 2)$

Since $P(x)$ passes through $(4, 54)$

$$54 = a(9)(6)$$

$$\therefore a = 1$$

$$\therefore P(x) = (x - 1)^2(x + 2)$$

4 **a** $y = 2(x - 1)(x + 2)(x + 4)$

has x -intercepts $1, -2, -4$

$$\text{has } y\text{-intercept } 2(-1)(2)(4) = -16$$

\therefore matches graph F

c $y = (x - 1)(x - 2)(x + 4)$

has x -intercepts $1, 2, -4$

$$\text{has } y\text{-intercept } (-1)(-2)(4) = 8$$

\therefore matches graph A

- e** $y = -(x - 1)(x + 2)(x + 4)$
has x -intercepts $1, -2, -4$
has y -intercept $-(1)(2)(4) = 8$
 \therefore matches graph D

- f** $y = 2(x - 1)(x - 2)(x + 4)$
has x -intercepts $1, 2, -4$
has y -intercept $2(-1)(-2)(4) = 16$
 \therefore matches graph B

- 5** **a** $\frac{1}{2}$ and -3 are zeros, and so $(2x - 1)$ and $(x + 3)$ are factors

$$\therefore P(x) = (2x - 1)(x + 3)(ax + b)$$

$$\text{But } P(0) = 30 \quad \therefore b(-1)(3) = 30 \quad \text{and so } b = -10$$

$$\therefore P(x) = (2x - 1)(x + 3)(ax - 10)$$

$$\text{Now } P(1) = (1)(4)(a - 10) = -20$$

$$\therefore a - 10 = -5 \quad \text{and so } a = 5$$

$$\therefore P(x) = (2x - 1)(x + 3)(5x - 10)$$

$$\therefore P(x) = 5(x - 2)(2x - 1)(x + 3)$$

- b** 1 is a zero and so $(x - 1)$ is a factor, touches at -2 indicates that $(x + 2)^2$ is a factor

$$\therefore P(x) = k(x - 1)(x + 2)^2$$

$$\text{But } P(0) = 8 \quad \therefore 8 = k(-1)(2)^2 \quad \text{and so } k = -2$$

$$\therefore P(x) = -2(x - 1)(x + 2)^2$$

- c** cuts the x -axis at $(2, 0)$ and so $(x - 2)$ is a factor

$$\therefore P(x) = (x - 2)(ax^2 + bx + c)$$

$$\text{But } P(0) = -4 \quad \therefore -2c = -4 \quad \text{and so } c = 2$$

$$\text{Also } P(1) = -1 \quad \therefore (-1)(a + b + 2) = -1$$

$$\therefore a + b + 2 = 1 \quad \therefore a + b = -1 \dots\dots (1)$$

$$\text{Also } P(-1) = -21 \quad \therefore (-3)(a - b + 2) = -21$$

$$\therefore a - b + 2 = 7 \quad \therefore a - b = 5 \dots\dots (2)$$

$$\text{Adding (1) and (2) gives } 2a = 4$$

$$\therefore a = 2 \quad \text{and so } b = -3$$

$$\therefore P(x) = (x - 2)(2x^2 - 3x + 2)$$

EXERCISE 7E.2

- b** $P(x) = ax(x + 2)(2x - 1)$
Since $P(x)$ passes through $(-3, -21)$

$$-21 = -3a(-1)(-7)$$

$$\therefore -21 = -21a$$

$$\therefore a = 1$$

$$\therefore P(x) = x(x + 2)(2x - 1)$$

- d** $P(x) = a(3x + 2)^2(x - 4)$
Since $P(x)$ passes through $(-1, -5)$

$$-5 = a(1)(-5)$$

$$\therefore a = 1$$

$$\therefore P(x) = (3x + 2)^2(x - 4)$$

- b** $y = -(x + 1)(x - 2)(x - 4)$
has x -intercepts $-1, 2, 4$
has y -intercept $-(1)(-2)(-4) = -8$

\therefore matches graph C

- d** $y = -2(x - 1)(x + 2)(x + 4)$
has x -intercepts $1, -2, -4$
has y -intercept $-2(-1)(2)(4) = 16$

\therefore matches graph E

- 1** **a** $P(x) = a(x + 1)^2(x - 1)^2$
where $a \neq 0$, and passes through $(0, 2)$

$$2 = a(1)(1)$$

$$\therefore a = 2$$

$$\therefore P(x) = 2(x + 1)^2(x - 1)^2$$

- c** $P(x) = a(x + 2)(x + 1)(x - 2)^2$
where $a \neq 0$, and passes through $(0, -16)$

$$-16 = a(2)(1)(4)$$

$$\therefore a = -2$$

$$\therefore P(x) = -2(x + 2)(x + 1)(x - 2)^2$$

- e** $P(x) = a(x + 1)(x - 4)^3$
where $a \neq 0$, and passes through $(0, -16)$

$$-16 = a(1)(-4)^3$$

$$\therefore a = \frac{1}{4}$$

$$\therefore P(x) = \frac{1}{4}(x + 1)(x - 4)^3$$

- b** $P(x) = a(x + 3)(x + 1)^2(3x - 2)$
where $a \neq 0$, and passes through $(0, -6)$

$$-6 = a(3)(1)(-2)$$

$$\therefore a = 1$$

$$\therefore P(x) = (x + 3)(x + 1)^2(3x - 2)$$

- d** $P(x) = a(x + 3)(x + 1)(2x - 3)(x - 3)$
where $a \neq 0$, and passes through $(0, -9)$

$$-9 = a(3)(1)(-3)(-3)$$

$$\therefore a = -\frac{1}{3}$$

$$\therefore P(x) = -\frac{1}{3}(x + 3)(x + 1)(2x - 3)(x - 3)$$

- f** $P(x) = ax^2(x + 2)(x - 3)$
where $a \neq 0$, and passes through $(-3, 54)$

$$54 = a(9)(-1)(-6)$$

$$\therefore 54 = 54a$$

$$\therefore a = 1$$

$$\therefore P(x) = x^2(x + 2)(x - 3)$$

- 2** **a** $y = (x - 1)^2(x + 1)(x + 3)$
has x -intercepts $-1, -3$, touches at 1
has y -intercept $(-1)^2(1)(3) = 3 (> 0)$
 \therefore matches graph C
- c** $y = (x - 1)(x + 1)^2(x + 3)$
has x -intercepts $1, -3$, touches at -1
has y -intercept $(-1)(1)^2(3) = -3 (< 0)$
 \therefore matches graph A
- e** $y = -\frac{1}{3}(x - 1)(x + 1)(x + 3)^2$
has x -intercepts $1, -1$, touches at -3
has y -intercept $-\frac{1}{3}(-1)(1)(3)^2 = 3 (> 0)$
 \therefore matches graph B

3 **a** $P(x) = a(x + 4)(2x - 1)(x - 2)^2$
where $a \neq 0$, and passes through $(1, 5)$
 $5 = a \times 5 \times 1 \times 1$
 $\therefore a = 1$
 $\therefore P(x) = (x + 4)(2x - 1)(x - 2)^2$

b $P(x) = a(3x - 2)^2(x + 3)^2$
where $a \neq 0$, and passes through $(-4, 49)$
 $49 = a(-14)^2(1)$
 $\therefore a = \frac{1}{4}$
 $\therefore P(x) = \frac{1}{4}(3x - 2)^2(x + 3)^2$

c $P(x) = a(2x + 1)(2x - 1)(x + 2)(x - 2)$
where $a \neq 0$, and passes through $(1, -18)$
 $-18 = a(3)(1)(3)(-1)$
 $\therefore a = 2$
 $\therefore P(x) = 2(2x + 1)(2x - 1)(x + 2)(x - 2)$

EXERCISE 7E.3

- 1** **a** $P(x) = x^3 - 3x^2 - 3x + 1$
From technology, -1 is a zero.
Check: $P(-1) = -1 - 3 + 3 + 1 = 0 \checkmark$
 $\therefore x + 1$ is a factor
 $\therefore x^3 - 3x^2 - 3x + 1 = (x + 1)(x^2 - 4x + 1)$
and the quadratic has zeros of $\frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$
 \therefore zeros are $-1, 2 \pm \sqrt{3}$

- b** $P(x) = x^3 - 3x^2 + 4x - 2$
From technology, 1 is a zero
Check: $P(1) = 1 - 3 + 4 - 2 = 0 \checkmark$
 $\therefore x - 1$ is a factor
From the division process $x^2 - 2x + 2$ is a quadratic factor
and it has zeros of $\frac{2 \pm \sqrt{4 - 4 \times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$
 \therefore zeros are $1, 1 \pm i$

- b** $y = -2(x - 1)^2(x + 1)(x + 3)$
has x -intercepts $-1, -3$, touches at 1
has y -intercept $-2(-1)^2(1)(3) = -6 (< 0)$
 \therefore matches graph F
- d** $y = (x - 1)(x + 1)^2(x - 3)$
has x -intercepts $1, 3$, touches at -1
has y -intercept $(-1)(1)^2(-3) = 3 (> 0)$
 \therefore matches graph E
- f** $y = -(x - 1)(x + 1)(x - 3)^2$
has x -intercepts $1, -1$, touches at 3
has y -intercept $-(1)(1)^2(3)^2 = 9 (> 0)$
 \therefore matches graph D

d $P(x) = (x - 1)^2(ax^2 + bx + c)$
where $a \neq 0$, and cuts y -axis at $(0, -1)$
 $-1 = 1 \times (0 + 0 + c)$
 $\therefore c = -1$
 $\therefore P(x) = (x - 1)^2(ax^2 + bx - 1)$
But $P(-1) = -4$
 $\therefore -4 = 4(a - b - 1)$
 $\therefore a - b = 0 \dots\dots(1)$
Also $P(2) = 15$
 $\therefore 15 = 1(4a + 2b - 1)$
 $\therefore 16 = 4a + 2b$
 $\therefore 2a + b = 8 \dots\dots(2)$
Adding (1) and (2) we get:
 $\therefore a = \frac{8}{3}$ and so $b = \frac{8}{3}$ also
 $\therefore P(x) = (x - 1)^2(\frac{8}{3}x^2 + \frac{8}{3}x - 1)$

-1	1	-3	-3	1
	0	-1	4	-1
1	-4	1		0

1	1	-3	4	-2
	0	1	-2	2
1	-2	2		0

c $P(x) = 2x^3 - 3x^2 - 4x - 35$

From technology, $\frac{7}{2}$ is a zero.

$$\begin{aligned} \text{Check: } P\left(\frac{7}{2}\right) &= \frac{343}{4} - \frac{147}{4} - 14 - 35 \\ &= \frac{343 - 147 - 56 - 140}{4} \\ &= 0 \quad \checkmark \end{aligned}$$

From the division process $2x^2 + 4x + 10$ is a quadratic factor

$$\begin{aligned} \therefore P(x) &= (x - \frac{7}{2})(2x^2 + 4x + 10) \\ &= (2x - 7)(x^2 + 2x + 5) \end{aligned}$$

$$\text{where the quadratic has zeros } \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

\therefore zeros are $\frac{7}{2}, -1 \pm 2i$

d $P(x) = 2x^3 - x^2 + 20x - 10$

From technology, $\frac{1}{2}$ is a zero.

$$\text{Check: } P\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{4} + 10 - 10 = 0 \quad \checkmark$$

$$\begin{aligned} \therefore P(x) &= (x - \frac{1}{2})(2x^2 + 20) \\ &= (2x - 1)(x^2 + 10) \end{aligned}$$

\therefore zeros are $\frac{1}{2}, \pm i\sqrt{10}$

e $P(x) = 4x^4 - 4x^3 - 25x^2 + x + 6$

From technology, -2 and 3 are zeros

$$\begin{aligned} \text{Check: } P(-2) &= 64 + 32 - 100 - 2 + 6 = 0 \quad \checkmark \\ P(3) &= 324 - 108 - 225 + 3 + 6 = 0 \quad \checkmark \end{aligned}$$

$$\therefore P(x) = (x + 2)(x - 3)(4x^2 - 1)$$

\therefore zeros are $-2, 3, \pm \frac{1}{2}$

f $P(x) = x^4 - 6x^3 + 22x^2 - 48x + 40$

From technology, 2 seems to be a double zero.

{Graph touches the x -axis at 2}

$$\text{Check: } P(2) = 16 - 48 + 88 - 96 + 40 = 0 \quad \checkmark$$

$$\therefore P(x) = (x - 2)^2(x^2 - 2x + 10)$$

$$\text{where the quadratic has zeros of } \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm 3i$$

\therefore zeros are $2, 2, 1 \pm 3i$

g $P(x) = x^3 + 2x^2 + 3x + 6$

From technology, -2 is a zero.

$$\text{Check: } P(-2) = -8 + 8 - 6 + 6 = 0 \quad \checkmark$$

$$\therefore P(x) = (x + 2)(x^2 + 3)$$

$$= (x + 2)(x + i\sqrt{3})(x - i\sqrt{3})$$

\therefore roots of $P(x) = 0$ are $x = -2$ and $x = \pm i\sqrt{3}$

h $P(x) = 2x^3 + 3x^2 - 3x - 2$

From technology, 1 is a zero.

$$\text{Check: } P(1) = 2 + 3 - 3 - 2 = 0 \quad \checkmark$$

$$\therefore P(x) = (x - 1)(2x^2 + 5x + 2)$$

$$= (x - 1)(2x + 1)(x + 2)$$

\therefore roots of $P(x) = 0$ are $1, -\frac{1}{2}, -2$

$$\begin{array}{r|rrrr} \frac{7}{2} & 2 & -3 & -4 & -35 \\ \hline & 0 & 7 & 14 & 35 \\ & 2 & 4 & 10 & 0 \end{array}$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & 20 & -10 \\ \hline & 0 & 1 & 0 & 10 \\ & 2 & 0 & 20 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 4 & -4 & -25 & 1 & 6 \\ \hline & 0 & -8 & 24 & 2 & -6 \\ & 4 & -12 & -1 & 3 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 22 & -48 & 40 \\ \hline & 0 & 2 & -8 & 28 & -40 \\ & 1 & -4 & 14 & -20 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 3 & 6 \\ \hline & 0 & -2 & 0 & -6 \\ & 1 & 0 & 3 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -3 & -2 \\ \hline & 0 & 2 & 5 & 2 \\ & 2 & 5 & 2 & 0 \end{array}$$

c $P(x) = x^3 - 6x^2 + 12x - 8$
 From technology, 2 is a zero.
 Check: $P(2) = 8 - 24 + 24 - 8 = 0 \quad \checkmark$
 $\therefore P(x) = (x-2)(x^2 - 4x + 4)$
 $= (x-2)(x-2)(x-2)$
 \therefore only root of $P(x) = 0$ is $x = 2$ (a treble root)

2	1	-6	12	-8	
	0	2	-8	8	
	1	-4	4		

d $P(x) = 2x^3 - 5x^2 - 9x + 18$
 From technology, 3 is a zero.
 Check: $P(3) = 54 - 45 - 27 + 18 = 0 \quad \checkmark$
 $\therefore P(x) = (x-3)(2x^2 + x - 6)$
 $= (x-3)(2x-3)(x+2)$
 \therefore roots of $P(x) = 0$ are 3, $\frac{3}{2}$ and -2

3	2	-5	-9	18	
	0	6	3	-18	
	2	1	-6		

e $P(x) = x^4 - x^3 - 9x^2 + 11x + 6$
 From technology, 2 and -3 are zeros.
 Check: $P(2) = 16 - 8 - 36 + 22 + 6 = 0 \quad \checkmark$
 $P(-3) = 81 + 27 - 81 - 33 + 6 = 0 \quad \checkmark$
 $\therefore P(x) = (x-2)(x+3)(x^2 - 2x - 1)$
 $\text{where the quadratic has zeros of } \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

2	1	-1	-9	11	6	
	0	2	2	-14	-6	
	1	1	-7	-3	0	
	0	-3	6	3		

f $P(x) = 2x^4 - 13x^3 + 27x^2 - 13x - 15$
 From technology, $-\frac{1}{2}$ and 3 are zeros.
 Check: $P\left(-\frac{1}{2}\right) = \frac{1}{8} + \frac{13}{8} + \frac{27}{4} + \frac{13}{2} - 15 = 0 \quad \checkmark$
 $P(3) = 162 - 351 + 243 - 39 - 15 = 0 \quad \checkmark$
 $\therefore P(x) = (x + \frac{1}{2})(x-3)(2x^2 - 8x + 10)$
 $= (2x+1)(x-3)(x^2 - 4x + 5)$
 $\text{where the quadratic has zeros of } \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$
 \therefore roots of $P(x) = 0$ are $-\frac{1}{2}, 3, 2 \pm i$

-1/2	2	-13	27	-13	-15	
	0	-1	7	-17	15	
	2	-14	34	-30		
	0	6	-24	30		

3 a Consider $P(x) = x^3 - 3x^2 + 4x - 2$
 From technology, 1 is a zero.
 Check: $P(1) = 1 - 3 + 4 - 2 = 0 \quad \checkmark$
 Now $P(x) = (x-1)(x^2 - 2x + 2)$
 $\text{where the quadratic has zeros of } \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$
 $\therefore P(x) = (x-1)(x-[1+i])(x-[1-i])$

1	1	-3	4	-2	
	0	1	-2	2	
	1	-2	2		

b Consider $P(x) = x^3 + 3x^2 + 4x + 12$
 From technology, -3 is a zero.
 Check: $P(-3) = -27 + 27 - 12 + 12 = 0 \quad \checkmark$
 Now $P(x) = (x+3)(x^2 + 4)$
 $\therefore P(x) = (x+3)(x-2i)(x+2i)$

-3	1	3	4	12	
	0	-3	0	-12	
	1	0	4		

c Consider $P(x) = 2x^3 - 9x^2 + 6x - 1$
 From technology, $\frac{1}{2}$ is a zero.
 Check: $P(\frac{1}{2}) = \frac{1}{4} - \frac{9}{4} + 3 - 1 = 0 \quad \checkmark$

1/2	2	-9	6	-1	
	0	1	-4	1	
	2	-8	2		

Now $P(x) = (x - \frac{1}{2})(2x^2 - 8x + 2)$

$= (2x-1)(x^2 - 4x + 1)$

where the quadratic has zeros of $\frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\therefore P(x) = (2x-1)(x-[2+\sqrt{3}])(x-[2-\sqrt{3}])$

d $P(x) = x^3 - 4x^2 + 9x - 10$

From technology, 2 is a zero.

Check: $P(2) = 8 - 16 + 18 - 10 = 0 \quad \checkmark$

Now $P(x) = (x-2)(x^2 - 2x + 5)$

where the quadratic has zeros of $\frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\therefore P(x) = (x-2)(x-[1+2i])(x-[1-2i])$

e $P(x) = 4x^3 - 8x^2 + x + 3$

From technology, 1 is a zero.

Check: $P(1) = 4 - 8 + 1 + 3 = 0 \quad \checkmark$

Now $P(x) = (x-1)(4x^2 - 4x - 3)$

$= (x-1)(2x-3)(2x+1)$

$\therefore P(x) = (x-1)(2x+1)(2x-3)$

f $P(x) = 3x^4 + 4x^3 + 5x^2 + 12x - 12$

From technology, -2 and $\frac{2}{3}$ are zeros.

Check: $P(-2) = 48 - 32 + 20 - 24 - 12 = 0 \quad \checkmark$

$P\left(\frac{2}{3}\right) = \frac{16}{27} + \frac{32}{27} + \frac{20}{9} + 8 - 12 = 0 \quad \checkmark$

Now $P(x) = (x+2)(x-\frac{2}{3})(3x^2 + 9)$

$= (x+2)(3x-2)(x^2 + 3)$

$\therefore P(x) = (x+2)(3x-2)(x+i\sqrt{3})(x-i\sqrt{3})$

g $P(x) = 2x^4 - 3x^3 + 5x^2 + 6x - 4$

From technology, -1 and $\frac{1}{2}$ are zeros.

Check: $P(-1) = 2 + 3 + 5 - 6 - 4 = 0 \quad \checkmark$

$P\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{3}{8} + \frac{5}{4} + 3 - 4 = 0 \quad \checkmark$

Now $P(x) = (x+1)(x-\frac{1}{2})(2x^2 - 4x + 8)$

$= (x+1)(2x-1)(x^2 - 2x + 4)$

where the quadratic has zeros of $\frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$

$\therefore P(x) = (x+1)(2x-1)(x-[1+i\sqrt{3}])(x-[1-i\sqrt{3}])$

h $P(x) = 2x^3 + 5x^2 + 8x + 20$

From technology, $-\frac{5}{2}$ is a zero.

Check: $P(-\frac{5}{2}) = -\frac{125}{8} + \frac{125}{4} - 20 + 20 = 0 \quad \checkmark$

Now $P(x) = (x + \frac{5}{2})(2x^2 + 8)$

$= (2x+5)(x^2 + 4)$

$\therefore P(x) = (2x+5)(x-2i)(x+2i)$

* Using technology, $x^3 + 2x^2 - 6x - 6$ has zeros of -0.860, 2.133 and -3.273

* Using technology, $x^3 + x^2 - 7x - 8$ has zeros of -2.518, -1.178 and 2.696

EXERCISE 7F

1 Since it is a real polynomial, the zeros must be $-\frac{1}{2}$, $1 - 3i$ and $1 + 3i$.

For $1 \pm 3i$, $\alpha + \beta = 2$ and $\alpha\beta = 1 - 9i^2 = 10$
 \therefore factors are $(2x + 1)$ and $(x^2 - 2x + 10)$
 $\therefore P(x) = a(2x + 1)(x^2 - 2x + 10)$, $a \neq 0$

$$2 p(1) = p(2 + i) = 0$$

Hence zeros of $p(x)$ must be $1, 2 \pm i$ {as $p(x)$ is real}

For $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

\therefore factors must be $(x - 1)$ and $(x^2 - 4x + 5)$

$$\therefore p(x) = k(x - 1)(x^2 - 4x + 5)$$

Since $p(0) = -20$ then $-20 = k(-1)(5)$

$$\therefore k = 4$$

$$\therefore p(x) = 4(x - 1)(x^2 - 4x + 5)$$

$$\therefore p(x) = 4x^3 - 20x^2 + 36x - 20$$

3 $2 - 3i$ is a zero of $z^3 + pz + q$ and as the cubic has real coefficients, $2 + 3i$ is also a zero.

For $2 \pm 3i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - 9i^2 = 13$

$\therefore z^2 - 4z + 13$ is a factor

$$\therefore z^3 + pz + q = (z^2 - 4z + 13)(z + a)$$
 for some a .

Equating coefficients:

$$a - 4 = 0, \quad 13 - 4a = p \quad \text{and} \quad 13a = q$$

$$\therefore a = 4, \quad p = -3, \quad q = 52$$

\therefore the other zeros are -4 and $2 + 3i$

$$\text{Check: Since } P(2 - 3i) = 0, \quad (2 - 3i)^3 + p(2 - 3i) + q = 0$$

Expanding, $(-46 - 9i) + p(2 - 3i) + q = 0$

$$\therefore (-46 + 2p + q) + (-9 - 3p)i = 0$$

Equating real and imaginary parts, $-46 + 2p + q = 0$ (1)

$$\text{and } -9 - 3p = 0 \quad \dots \dots (2)$$

From (2), $p = -3$, so in (1), $-46 - 6 + q = 0 \quad \therefore p = -3, q = 52 \quad \checkmark$

4 $3 + i$ is a root of $z^4 - 2z^3 + az^2 + bz + 10 = 0$ where the coefficients are real.

$\therefore 3 - i$ is also a root

For $3 \pm i$, $\alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore z^2 - 6z + 10$ is a factor

$$\therefore z^4 - 2z^3 + az^2 + bz + 10$$

$$= (z^2 - 6z + 10)(z^2 + sz + 1) \text{ for some } s.$$

Equating coefficients:

$$s - 6 = -2, \quad 11 - 6s = a \quad \text{and} \quad 10s - 6 = b$$

$$\therefore s = 4 \quad a = 11 - 6(4) = -13 \quad b = 10(4) - 6 = 34$$

$$\therefore \text{the other factor is } z^2 + 4z + 1 \text{ which has zeros } \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$$

$\therefore a = -13, b = 34$ and the other roots are $3 - i, -2 \pm \sqrt{3}$.

5 Let the purely imaginary zero be bi . Since $P(z)$ is real, another zero is $-bi$, (b is real)

$\therefore z^2 + b^2$ is a factor of $P(z)$

$$\therefore z^3 + az^2 + 3z + 9 = (z^2 + b^2)(z + c)$$

$$= z^3 + cz^2 + b^2z + b^2c$$

Equating coefficients, $b^2 = 3$, $b^2c = 9$ and $a = c$

$$\therefore c = 3 \quad \text{and} \quad a = 3 \quad \text{and} \quad b = \pm\sqrt{3}$$

$$\therefore P(z) = (z + 3)(z^2 + 3)$$

$$\therefore P(z) = (z + 3)(z + i\sqrt{3})(z - i\sqrt{3}), \quad a = 3$$

6 Let ai be the purely imaginary zero of $3x^3 + kx^2 + 15x + 10$

\therefore as $P(x)$ is real, $-ai$ is also a zero

For $\pm ai$, $\alpha + \beta = 0$ and $\alpha\beta = -a^2i^2 = a^2$

$\therefore x^2 + a^2$ is a factor

$$\therefore 3x^3 + kx^2 + 15x + 10 = (x^2 + a^2)(3x + b)$$

$$= 3x^3 + bx^2 + 3a^2x + a^2b$$

$$\begin{array}{r} 1 & 0 & a^2 \\ \times & 3 & b \\ \hline b & 0 & a^2b \\ \hline 3 & 0 & 3a^2 \\ \hline 3 & b & 3a^2 & a^2b \end{array}$$

Equating coefficients $k = b$

$$\text{and } 3a^2 = 15 \quad \therefore a^2 = 5$$

$$\text{and } a^2b = 10 \quad \therefore b = 2 \quad \therefore k = 2$$

$$\therefore P(x) = (x^2 + 5)(3x + 2)$$

$$\therefore P(x) = (3x + 2)(x - i\sqrt{5})(x + i\sqrt{5}), \quad k = 2$$

7 a $f(t) = kt(t - a)^2$

From the graph a is the t -value at the point where the graph touches the t -axis.

$$\therefore a = 700 \text{ milliseconds}$$

This represents the time when the barrier has returned to its original position.

b when $t = 100$ ms $f(t) = 85$ mm

$$\therefore 85 = k \times 100(100 - 700)^2$$

$$85 = 100 \times k \times 360000$$

$$k = \frac{85}{3600000}$$

$$\therefore f(t) = \frac{85t}{3600000}(t - 700)^2$$

c Using technology to find the maximum, on $0 \leq t \leq 700$, the maximum occurs when $t \approx 233$ ms
 \therefore when $f(t) \approx 120$ mm

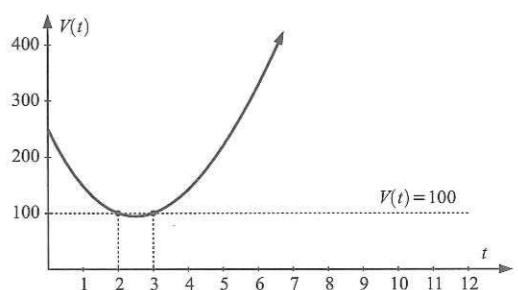
d $V(t) = -t^3 + 30t^2 - 131t + 250$

We graph $V(t)$ against t and add the graph of $V(t) = 100$.

From the graph, the level drops below 100 mL when $t = 2$ and rises above 100 mL again when $t = 3$.

Now as $t = 0$ is Jan 1st,
 $0 \leq t < 1$ is January.

\therefore as irrigation is prohibited for
 $2 < t < 3$, it is banned during March.



Let the height of the wall where the ladder touches be x m.

Using similar triangles AXB, AOC:

$$\frac{x-1}{1} = \frac{x}{OC}$$

$$\therefore OC = \frac{x}{x-1}$$

$$\text{but } x^2 + OC^2 = 10^2$$

$$x^2 + \left(\frac{x}{x-1}\right)^2 = 100$$

Using technology to find the intersection of $y = x^2 + \left(\frac{x}{x-1}\right)^2$ and $y = 100$

$$x \approx 1.112 \text{ or } 9.938$$

So, distance ≈ 9.94 m or 1.11 m

