Chapter 13

LINES AND PLANES IN SPACE

EXERCISE 13A.1

1 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}$

ii $x = 3 + t, \quad y = -4 + 4t$

$\therefore \quad t = x - 3 = \dfrac{y+4}{4}$

$\therefore \quad 4x - 12 = y + 4$

$\therefore \quad 4x - y = 16$

b **i** If the line has direction vector $\mathbf{b}$ perpendicular to $\begin{pmatrix} -8 \\ 2 \end{pmatrix}$, then

$\mathbf{b} \bullet \begin{pmatrix} -8 \\ 2 \end{pmatrix} = 0$

$\therefore \quad \mathbf{b} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ is a reasonable choice

$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 8 \end{pmatrix}, \quad t \in \mathbb{R}$

ii $x = 5 + 2t, \quad y = 2 + 8t$

$\therefore \quad t = \dfrac{x-5}{2} = \dfrac{y-2}{8}$

$\therefore \quad 8x - 40 = 2y - 4$

$\therefore \quad 8x - 2y = 36$

$\therefore \quad 4x - y = 18$

c **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \end{pmatrix}, \quad t \in \mathbb{R}$

ii $x = -6 + 3t, \quad y = 7t$

$\therefore \quad t = \dfrac{x+6}{3} = \dfrac{y}{7}$

$\therefore \quad 7x + 42 = 3y$

$\therefore \quad 7x - 3y = -42$

d **i** Take $(-1, 11)$ as our fixed point,

so $\mathbf{a} = \begin{pmatrix} -1 \\ 11 \end{pmatrix}$.

The direction vector $\mathbf{b} = \begin{pmatrix} -3-(-1) \\ 12-11 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$

ii $x = -1 - 2t, \quad y = 11 + t$

$\therefore \quad t = \dfrac{x+1}{-2} = y - 11$

$\therefore \quad x + 1 = -2y + 22$

$\therefore \quad x + 2y = 21$

2 $x = -1 + 2t, \ y = 4 - t, \quad t \in \mathbb{R}$

When $t = 0$, $x = -1 + 2(0) = -1$ and $y = 4 - 0 = 4$ $\quad \therefore$ the point is $(-1, 4)$.

When $t = 1$, $x = -1 + 2(1) = 1$ and $y = 4 - 1 = 3$ $\quad \therefore$ the point is $(1, 3)$.

When $t = 3$, $x = -1 + 2(3) = 5$ and $y = 4 - 3 = 1$ $\quad \therefore$ the point is $(5, 1)$.

When $t = -1$, $x = -1 + 2(-1) = -3$ and $y = 4 - -1 = 5$ $\quad \therefore$ the point is $(-3, 5)$.

When $t = -4$, $x = -1 + 2(-4) = -9$ and $y = 4 - -4 = 8$ $\quad \therefore$ the point is $(-9, 8)$.

3 **a** If $t + 2 = 3$ and $1 - 3t = -2$,

then $t = 1$ and $-3t = -3$

$\therefore \quad t = 1$

Since $t = 1$ in each case, $(3, -2)$ lies on the line.

b If $(k, 4)$ lies on $x = 1 - 2t, \quad y = 1 + t$, then

$k = 1 - 2t$ and $4 = 1 + t$

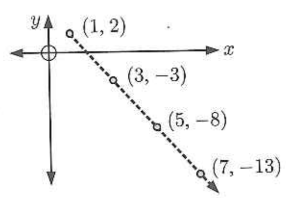
$\therefore \quad t = 3$

and $k = 1 - 6 = -5$

4 **a** $x(0) = 1$ and $y(0) = 2$,

$\therefore$ the initial position is $(1, 2)$

b



c In 1 second, the x-step is 2 and y-step is -5, which is a distance of $\sqrt{2^2 + (-5)^2} = \sqrt{29}$ cm

$\therefore$ the speed is $\sqrt{29}$ cm s^{-1}.

5 In parametric form: $\quad x = 1 - t, \quad y = 5 + 3t, \quad t \in \mathbb{R}$

In Cartesian form: $\quad t = \dfrac{x-1}{-1} = \dfrac{y-5}{3}$

$\therefore \quad 3x - 3 = -y + 5$

$\therefore \quad 3x + y = 8$

EXERCISE 13A.2

1 **a** The vector equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$

b The vector equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad t \in \mathbb{R}$

c Since the line is parallel to the X-axis, it has direction vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ the vector equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$

2 **a** The parametric equations are:

$x = 5 + (-1)t, \ y = 2 + 2t, \ z = -1 + 6t$

$\therefore \quad x = 5 - t, \ y = 2 + 2t, \ z = -1 + 6t, \ t \in \mathbb{R}$

b The parametric equations are:

$x = 0 + 2t, \ y = 2 + (-1)t, \ z = -1 + 3t$

$\therefore \quad x = 2t, \ y = 2 - t, \ z = -1 + 3t, \ t \in \mathbb{R}$

c Since the line is perpendicular to the XOY plane, it has direction vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\therefore$ the parametric equations are: $\quad x = 3 + 0t, \ y = 2 + 0t, \ z = -1 + 1t$

$\therefore \quad x = 3, \ y = 2, \ z = -1 + t, \ t \in \mathbb{R}$

3 **a** $\overrightarrow{AB} = \begin{pmatrix} -1-1 \\ 3-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$

b $\overrightarrow{CD} = \begin{pmatrix} 3-0 \\ 1-1 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \quad t \in \mathbb{R}$

c $\overrightarrow{EF} = \begin{pmatrix} 1-1 \\ -1-2 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$

d $\overrightarrow{GH} = \begin{pmatrix} 5-0 \\ -1-1 \\ 3--1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}$

4 Given $x = 1 - t, \ y = 3 + t, \ z = 3 - 2t$:

a The line meets the XOY plane when $z = 0$ $\quad \therefore \quad 3 - 2t = 0$

$\therefore \quad t = \frac{3}{2}$

Then $x = 1 - \frac{3}{2} = -\frac{1}{2}$ and $y = 3 + \frac{3}{2} = \frac{9}{2}$, so the point is $\left(-\frac{1}{2}, \frac{9}{2}, 0\right)$.

b The line meets the YOZ plane when $x = 0$ $\quad \therefore \quad 1 - t = 0$

$\therefore \quad t = 1$

Then $y = 3 + 1 = 4$ and $z = 3 - 2 = 1$, so the point is $(0, 4, 1)$.

c The line meets the XOZ plane when $y = 0$ $\quad \therefore\ 3 + t = 0$

$\therefore\ t = -3$

Then $x = 1 - (-3) = 4$ and $z = 3 - 2(-3) = 9$, so the point is $(4, 0, 9)$.

5 Given a line with equations $x = 2 - t$, $y = 3 + 2t$ and $z = 1 + t$,

the distance to the point $(1, 0, -2)$ is $\sqrt{(2-t-1)^2 + (3+2t-0)^2 + (1+t+2)^2}$.

But this distance $= 5\sqrt{3}$ units

$$\begin{aligned}
\therefore\ \sqrt{(1-t)^2 + (3+2t)^2 + (t+3)^2} &= 5\sqrt{3} \\
\therefore\ (1-t)^2 + (3+2t)^2 + (t+3)^2 &= 75 \\
\therefore\ 1 - 2t + t^2 + 9 + 12t + 4t^2 + t^2 + 6t + 9 &= 75 \\
\therefore\ 6t^2 + 16t - 56 &= 0 \\
\therefore\ 3t^2 + 8t - 28 &= 0 \\
\therefore\ (3t+14)(t-2) &= 0 \\
\therefore\ t = -\tfrac{14}{3} &\text{ or } t = 2
\end{aligned}$$

When $t = 2$ the point is $(0, 7, 3)$, and when $t = -\frac{14}{3}$ the point is $\left(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3}\right)$.

EXERCISE 13A.3

1 L_1 has direction vector $\binom{4}{-3}$ and L_2 has direction vector $\binom{5}{4}$. If θ is the angle between them,

$$\cos\theta = \frac{\left|\binom{4}{-3} \bullet \binom{5}{4}\right|}{\sqrt{16+9}\sqrt{25+16}} = \frac{|20 + (-12)|}{\sqrt{25 \times 41}} = \frac{8}{\sqrt{25 \times 41}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{8}{\sqrt{25 \times 41}}\right) \approx 75.5^\circ$$

$\therefore$ the required angle measures 75.5°.

2 L_1 has direction vector $\binom{12}{5}$ and L_2 has direction vector $\binom{3}{-4}$. If θ is the angle between them,

$$\cos\theta = \frac{\left|\binom{12}{5} \bullet \binom{3}{-4}\right|}{\sqrt{144+25}\sqrt{9+16}} = \frac{|36 + (-20)|}{13 \times 5} = \frac{16}{65}$$

$$\therefore\ \theta = \cos^{-1}\left(\tfrac{16}{65}\right) \approx 75.7^\circ$$

3 Line 1 has direction vector $\binom{5}{-2}$ and line 2 has direction vector $\binom{4}{10}$

and $\binom{5}{-2} \bullet \binom{4}{10} = 20 + (-20) = 0$

$\therefore$ the lines are perpendicular.

4 **a** Line 1 has direction vector $\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$.

If θ is the angle between them,

$$\cos\theta = \frac{\left|\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}\right|}{\sqrt{9+256+49}\sqrt{9+64+25}} = \frac{|9 - 128 - 35|}{\sqrt{314}\sqrt{98}} = \frac{154}{\sqrt{314 \times 98}}$$

$\therefore\ \theta \approx 28.6^\circ$

b Since $L_1 \perp L_3$, $\quad \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -3 \\ x \end{pmatrix} = 0$

$$\therefore\ 48 + 7x = 0$$

$$\therefore\ x = -\tfrac{48}{7}$$

EXERCISE 13B.1

1 **a** **i** When $t = 0$, $\binom{x}{y} = \binom{-4}{3}$ $\therefore$ the object is at $(-4, 3)$.
ii The velocity vector is $\binom{12}{5}$.
iii The speed is $\sqrt{12^2 + 5^2} = 13\ \text{m s}^{-1}$

b **i** When $t = 0$, $\binom{x}{y} = \binom{0}{-6}$ $\therefore$ the object is at $(0, -6)$.
ii The velocity vector is $\binom{3}{-4}$.
iii The speed is $\sqrt{3^2 + (-4)^2} = 5\ \text{m s}^{-1}$

c **i** When $t = 0$, $\binom{x}{y} = \binom{-2}{-7}$ $\therefore$ the object is at $(-2, -7)$.
ii The velocity vector is $\binom{-6}{-4}$.
iii The speed is $\sqrt{(-6)^2 + (-4)^2} = \sqrt{52}\ \text{m s}^{-1}$

d **i** When $t = 0$, $x = 5$ and $y = -5$ $\therefore$ the object is at $(5, -5)$.
ii The velocity vector is $\binom{8}{4}$.
iii The speed is $\sqrt{8^2 + 4^2} = \sqrt{80}\ \text{m s}^{-1}$

2 **a** $\binom{4}{-3}$ has length $\sqrt{4^2 + (-3)^2} = 5$

$\therefore\ 30\binom{4}{-3}$ has length 150

$\therefore$ the velocity vector is $\binom{120}{-90}$.

b $\binom{24}{7}$ has length $\sqrt{24^2 + 7^2} = 25$

$\therefore\ \frac{1}{2}\binom{24}{7}$ has length 12.5

$\therefore$ the velocity vector is $\binom{12}{3.5}$.

c $2\mathbf{i} + \mathbf{j} = \binom{2}{1}$ has length $\sqrt{2^2 + 1^2} = \sqrt{5}$

$\therefore\ 10\sqrt{5}\binom{2}{1}$ has length 50

$\therefore$ the velocity vector is $\binom{20\sqrt{5}}{10\sqrt{5}}$.

3 $-2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}$ has length $\sqrt{(-2)^2 + 5^2 + (-14)^2} = \sqrt{4 + 25 + 196} = \sqrt{225} = 15$

$\therefore\ 6\begin{pmatrix} -2 \\ 5 \\ -14 \end{pmatrix}$ has length 90, so the velocity vector is $\begin{pmatrix} -12 \\ 30 \\ -84 \end{pmatrix}$.

EXERCISE 13B.2

1 **a** $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

$\therefore\ \binom{x}{y} = \binom{-3}{-2} + t\binom{2}{4}, \quad t \geqslant 0$

$\therefore\ \binom{x}{y} = \binom{-3+2t}{-2+4t}$

b At $t = 2.5$, $\quad -3 + 2t = -3 + 5 = 2$

and $\quad -2 + 4t = -2 + 10 = 8$

So, the position vector is $\binom{2}{8}$.

c **i** When the car is due north, $x = 0$.

$\therefore\ -3 + 2t = 0$

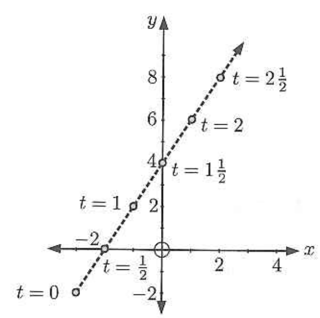
$\therefore\ t = 1.5$ seconds

ii When the car is due west, $y = 0$.

$\therefore\ -2 + 4t = 0$

$\therefore\ t = 0.5$ seconds

d



2 Yacht A: $\binom{x_A}{y_B} = \binom{4}{5} + t\binom{1}{-2}$ $\quad$ Yacht B: $\binom{x_B}{y_B} = \binom{1}{-8} + t\binom{2}{1}, \quad t \geqslant 0$

a When $t = 0$, $\binom{x_A}{y_A} = \binom{4}{5}$ $\quad \therefore$ A is at $(4, 5)$

and $\binom{x_B}{y_B} = \binom{1}{-8}$ $\quad \therefore$ B is at $(1, -8)$.

b For A, the velocity vector is $\binom{1}{-2}$, and for B it is $\binom{2}{1}$.

c Speed of A $= \sqrt{1^2 + (-2)^2} = \sqrt{5}\ \text{km h}^{-1}$. Speed of B $= \sqrt{2^2 + 1^2} = \sqrt{5}\ \text{km h}^{-1}$.

d A has direction vector $\begin{pmatrix}1\\-2\end{pmatrix}$ and B has direction vector $\begin{pmatrix}2\\1\end{pmatrix}$.

Since $\begin{pmatrix}1\\-2\end{pmatrix} \bullet \begin{pmatrix}2\\1\end{pmatrix} = 2-2 = 0$, the paths of the yachts are at right angles to each other.

3 **a** P's torpedo has position $\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}-5\\4\end{pmatrix} + t\begin{pmatrix}3\\-1\end{pmatrix}$ and at $t=0$, the time is 1:34 pm

$\therefore\ x_1(t) = -5+3t, \quad y_1(t) = 4-t.$

b Speed $= \sqrt{3^2+(-1)^2} = \sqrt{10}$ km min^{-1}

c Q fires its torpedo after a minutes.

$\therefore$ at time t, its torpedo has travelled for $(t-a)$ minutes.

$\therefore\ \begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}15\\7\end{pmatrix} + (t-a)\begin{pmatrix}-4\\-3\end{pmatrix}, \quad t \geqslant a$

$\therefore\ x_2(t) = 15-4(t-a)$ and $y_2(t) = 7-3(t-a)$

d They meet when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$

$\therefore\ -5+3t = 15-4(t-a)$ and $4-t = 7-3(t-a)$

$\therefore\ 7t-4a = 20$ .... (1) and $2t-3a = 3$ .... (2)

Solving simultaneously,

$$\begin{aligned} 21t - 12a &= 60 \quad \{3\times(1)\}\\ -8t+12a &= -12 \quad \{-4\times(2)\}\\ \text{adding}\quad 13t &= 48 \end{aligned}$$

$\therefore\ t = \frac{48}{13}$ and $7\left(\frac{48}{13}\right) - 4a = 20$

$\therefore\ t \approx 3.6923 \qquad \therefore\ 5.8462 = 4a$

$\therefore\ t \approx 3$ min 41.54 sec $\qquad \therefore\ a \approx 1.4615 \approx 1$ min 27.7 sec

So, as $a \approx 1.4615$, Q fired at 1:35:28 pm, and the explosion occurred at 1:37:42 pm.

4 **a** $\overrightarrow{AB} = \begin{pmatrix}3-6\\10-9\\2.5-3\end{pmatrix} = \begin{pmatrix}-3\\1\\-0.5\end{pmatrix}$

b $|\overrightarrow{AB}| = \sqrt{(-3)^2+1^2+(-0.5)^2} = \sqrt{10.25}$ km

The helicopter travels $\sqrt{10.25}$ km in 10 minutes.

$\therefore$ the helicopter's speed is $6 \times \sqrt{10.25} \approx 19.2$ km h^{-1}.

c $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6\\9\\3\end{pmatrix} + t\begin{pmatrix}-3\\1\\-0.5\end{pmatrix}, \quad t \in \mathbb{R}$

d If $z=0$, $3+(-0.5)t = 0$

$\therefore\ t = 6$

$t=1$ represents 10 minutes, so $t=6$ represents 60 minutes.

$\therefore$ the helicopter lands on the helipad after 1 hour.

EXERCISE 13B.3

1 **a** $6\mathbf{i} - 6\mathbf{j}$

b The length of $\begin{pmatrix}-3\\4\end{pmatrix} = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$

As the speed is 10 km h^{-1}, the liner has velocity vector $2\begin{pmatrix}-3\\4\end{pmatrix} = \begin{pmatrix}-6\\8\end{pmatrix}$.

$\therefore$ the liner has position $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}6\\-6\end{pmatrix} + t\begin{pmatrix}-6\\8\end{pmatrix}, \quad t \geqslant 0, \quad t$ in hours.

c The liner is due east when $y = 0$

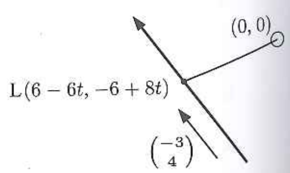
$\therefore\ -6+8t = 0$

$\therefore$ at $t = \frac{3}{4}$ hour

d The liner L is nearest the fishing boat O when $\overrightarrow{OL} \perp \begin{pmatrix}-3\\4\end{pmatrix}$.

$\therefore\ \overrightarrow{OL} \bullet \begin{pmatrix}-3\\4\end{pmatrix} = 0$

$\therefore\ \begin{pmatrix}6-6t\\-6+8t\end{pmatrix} \bullet \begin{pmatrix}-3\\4\end{pmatrix} = 0$



$\therefore\ (-18+18t) + (-24+32t) = 0$

$\therefore\ 50t = 42$

$\therefore\ t = 0.84$ hours $= 50.4$ minutes

and when $t = 0.84$, $\mathrm{L} = \begin{pmatrix}6-6(0.84)\\-6+8(0.84)\end{pmatrix} = \begin{pmatrix}0.96\\0.72\end{pmatrix}$

$\therefore$ the liner is closest to the fishing boat after 0.84 hours or 50.4 minutes, when it is at (0.96, 0.72).

2 **a** $|\mathbf{b}| = \sqrt{(-3)^2+(-1)^2} = \sqrt{10}$

As the speed is $40\sqrt{10}$ km h^{-1}, the velocity vector is $40\begin{pmatrix}-3\\-1\end{pmatrix} = \begin{pmatrix}-120\\-40\end{pmatrix}$.

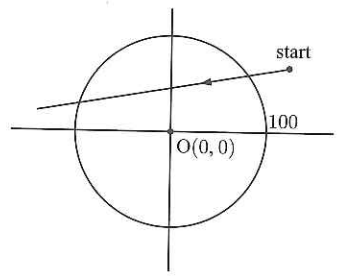
b $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}200\\100\end{pmatrix} + t\begin{pmatrix}-120\\-40\end{pmatrix}, \quad t \geqslant 0 \quad \{t = 0$ at 12:00 noon$\}$

c At 1:00 pm, $t = 1$ and $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}200-120\\100-40\end{pmatrix} = \begin{pmatrix}80\\60\end{pmatrix}$

d The distance from O(0, 0) to P$_1$(80, 60) is $\left|\begin{pmatrix}80\\60\end{pmatrix}\right| = \sqrt{80^2+60^2} = 100$ km,

which is when it becomes visible to radar. {within 100 km of O(0, 0)}

e



A general point on the path is P$(200-120t, 100-40t)$.

Now $\overrightarrow{OP} = \begin{pmatrix}200-120t\\100-40t\end{pmatrix}$,

and for the closest point $\overrightarrow{OP} \bullet \begin{pmatrix}-3\\-1\end{pmatrix} = 0$

$\therefore\ -3(200-120t) - 1(100-40t) = 0$

$\therefore\ -700+400t = 0$

$\therefore\ t = \frac{7}{4} = 1\frac{3}{4}$ hours

$\therefore$ the time when the aircraft is closest is 1:45 pm, and

at this time $\overrightarrow{OP} = \begin{pmatrix}200-120(\frac{7}{4})\\100-40(\frac{7}{4})\end{pmatrix} = \begin{pmatrix}-10\\30\end{pmatrix}$

$\therefore\ d_{\min} = \sqrt{(-10)^2+30^2} \approx 31.6$ km

f It disappears from radar when $|\overrightarrow{OP}| = 100$ and $t > 1\frac{3}{4}$

$\therefore\ \sqrt{(200-120t)^2+(100-40t)^2} = 100$

$\therefore\ 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1600t^2 = 10\,000$

$\therefore\ 16\,000t^2 - 56\,000t + 40\,000 = 0$

$\therefore\ 16t^2 - 56t + 40 = 0 \quad \{\div 1000\}$

$\therefore\ 2t^2 - 7t + 5 = 0 \quad \{\div 8\}$

$\therefore\ (2t-5)(t-1) = 0$

$\therefore\ t = \frac{5}{2} \quad \{\text{as } t > 1\frac{3}{4}\}$

So, the aircraft disappears from the radar screen $2\frac{1}{2}$ hours after noon, or at 2:30 pm.

3 **a** At A, $y = 0$

$\therefore\ 2x = 36$

$\therefore\ x = 18$

At B, $x = 0$

$\therefore\ 3y = 36$

$\therefore\ y = 12$

So A is (18, 0) and B is (0, 12).

b $2x+3y = 36$

$\therefore\ 3y = 36-2x$

$\therefore\ y = \dfrac{36-2x}{3}$

$\therefore$ any point R on the railway track can be written $\mathrm{R}\left(x, \dfrac{36-2x}{3}\right)$.

c $\overrightarrow{PR} = \begin{pmatrix}x-4\\\frac{36-2x}{3}-0\end{pmatrix} = \begin{pmatrix}x-4\\\frac{36-2x}{3}\end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix}0-18\\12-0\end{pmatrix} = \begin{pmatrix}-18\\12\end{pmatrix}$

d The point closest to the railway track is R such that $\overrightarrow{PR} \perp \overrightarrow{AB}$.

$$\therefore \quad \overrightarrow{PR} \bullet \overrightarrow{AB} = 0$$

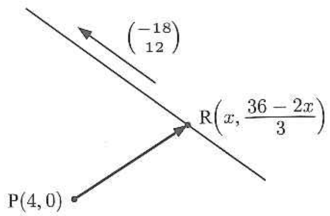
$$\therefore \quad \begin{pmatrix} x-4 \\ \frac{36-2x}{3} \end{pmatrix} \bullet \begin{pmatrix} -18 \\ 12 \end{pmatrix} = 0$$

$$\therefore \quad -18(x-4) + 4(36-2x) = 0$$

$$\therefore \quad -18x + 72 + 144 - 8x = 0$$

$$\therefore \quad 26x = 216$$

$$\therefore \quad x = \tfrac{108}{13}$$



Now when $x = \frac{108}{13}$, $\quad \dfrac{36-2x}{3} = 12 - \frac{2}{3}x = 12 - \frac{2}{3}\left(\frac{108}{13}\right) = \frac{84}{13}$

So R is $\left(\frac{108}{13}, \frac{84}{13}\right)$.

$$|\overrightarrow{PR}| = \sqrt{\left(\tfrac{108}{13} - 4\right)^2 + \left(\tfrac{84}{13} - 0\right)^2} = \sqrt{\tfrac{784}{13}} \approx 7.77 \text{ km}$$

The closest point on the track to the camp is $\left(\frac{108}{13}, \frac{84}{13}\right)$, a distance of 7.77 km.

4 For A, $x_A(t) = 3 - t$, $y_A(t) = 2t - 4$ For B, $x_B(t) = 4 - 3t$, $y_B(t) = 3 - 2t$

a When $t = 0$, $x_A(0) = 3$, $y_A(0) = -4$ and $x_B(0) = 4$, $y_B(0) = 3$

$\therefore$ A is at $(3, -4)$. $\therefore$ B is at $(4, 3)$.

b The velocity vector of A is $\binom{-1}{2}$ and the velocity vector of B is $\binom{-3}{-2}$.

c If the angle is θ, $\binom{-1}{2} \bullet \binom{-3}{-2} = \sqrt{1+4}\sqrt{9+4}\cos\theta$

$$\therefore \quad 3 - 4 = \sqrt{5}\sqrt{13}\cos\theta$$

$$\therefore \quad \frac{-1}{\sqrt{65}} = \cos\theta \quad \text{and so} \quad \theta \approx 97.1^\circ$$

d If D is the distance between them, then

$$D = \sqrt{[(4-3t)-(3-t)]^2 + [(3-2t)-(2t-4)]^2}$$
$$= \sqrt{[1-2t]^2 + [7-4t]^2}$$
$$= \sqrt{1 - 4t + 4t^2 + 49 - 56t + 16t^2}$$
$$= \sqrt{20t^2 - 60t + 50}$$

Under the square root we have a quadratic in t, so D is a minimum when $t = -\dfrac{b}{2a} = \frac{60}{40} = 1\frac{1}{2}$

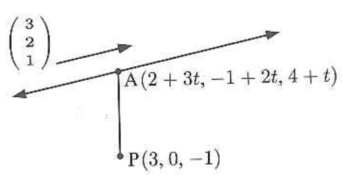
$\therefore \quad t = 1.5$ hours

5 **a** The direction vector of the line is $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Let the point $(3, 0, -1)$ be P, and $A(2+3t, -1+2t, 4+t)$ be any point on the line.

$$\therefore \quad \overrightarrow{PA} = \begin{pmatrix} 2+3t-3 \\ -1+2t-0 \\ 4+t-(-1) \end{pmatrix} = \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix}$$

Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$.



$$\therefore \quad \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\therefore \quad -3 + 9t - 2 + 4t + 5 + t = 0$$

$$\therefore \quad 14t = 0$$

$$\therefore \quad t = 0$$

Substituting $t = 0$ into the parametric equations, we obtain the foot of the perpendicular $(2, -1, 4)$.

b When $t = 0$, $\overrightarrow{PA} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$, so $PA = \sqrt{1+1+25}$
$= \sqrt{27}$ units

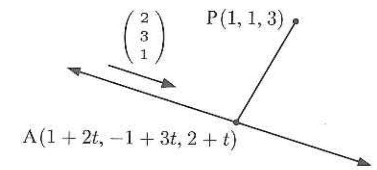
$\therefore$ the shortest distance from the point to the line is $3\sqrt{3}$ units.

6 **a** The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let the point $(1, 1, 3)$ be P and $A(1+2t, -1+3t, 2+t)$ be any point on the line.

$$\therefore \quad \overrightarrow{PA} = \begin{pmatrix} 1+2t-1 \\ -1+3t-1 \\ 2+t-3 \end{pmatrix} = \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix}.$$

Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$



$$\therefore \quad \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\therefore \quad 4t - 6 + 9t - 1 + t = 0$$

$$\therefore \quad 14t = 7$$

$$\therefore \quad t = \tfrac{1}{2}$$

Substituting $t = \frac{1}{2}$ into the parametric equations, we obtain the foot of the perpendicular $(2, \frac{1}{2}, \frac{5}{2})$.

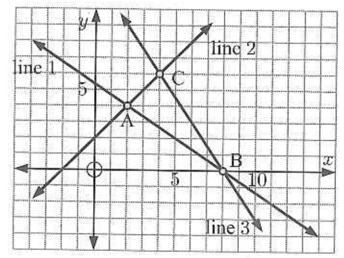
b When $t = \frac{1}{2}$, $\quad \overrightarrow{PA} = \begin{pmatrix} 1 \\ -2+\frac{3}{2} \\ -1+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$$\therefore \quad PA = \sqrt{1 + \tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{\tfrac{3}{2}} \text{ units}$$

$\therefore$ the shortest distance from the point to the line is $\sqrt{\frac{3}{2}}$ units.

EXERCISE 13B.4

1 **a**



b A is $(2, 4)$, B is $(8, 0)$, C is $(4, 6)$

c $BC = \sqrt{(4-8)^2 + (6-0)^2} = \sqrt{16+36} = \sqrt{52}$ units

$AB = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16} = \sqrt{52}$ units

$\therefore$ BC = AB and so ΔABC is isosceles.

d Line 1 and Line 2 meet at A.

$$\therefore \quad \binom{-1}{6} + \binom{3r}{-2r} = \binom{0}{2} + s\binom{1}{1}$$

$$\therefore \quad \binom{3r-s}{-2r-s} = \binom{1}{-4}$$

$\therefore \quad 3r - s = 1$

and $\quad 2r + s = 4$

Adding, $\quad 5r \quad = 5 \quad \therefore \quad r = 1$

$$\therefore \quad \binom{x}{y} = \binom{-1}{6} + \binom{3}{-2} = \binom{2}{4} \quad \checkmark$$

Line 2 and Line 3 meet at C.

$$\therefore \quad \binom{0}{2} + s\binom{1}{1} = \binom{10}{-3} + t\binom{-2}{3}$$

$$\therefore \quad \binom{s+2t}{s-3t} = \binom{10}{-5}$$

$\therefore \quad s + 2t = 10$

$\quad -s + 3t = 5$

Adding, $\quad 5t = 15 \quad \therefore \quad t = 3$

$$\therefore \quad \binom{x}{y} = \binom{10}{-3} + 3\binom{-2}{3} = \binom{4}{6} \quad \checkmark$$

Line 1 and Line 3 meet at B.

$\therefore \quad \binom{-1}{6} + r\binom{3}{-2} = \binom{10}{-3} + t\binom{-2}{3}$

$\therefore \quad \binom{3r+2t}{-2r-3t} = \binom{11}{-9}$

$\therefore \quad 3r + 2t = 11 \quad .... (1)$

$-2r - 3t = -9 \quad .... (2)$

$\therefore \quad 9r + 6t = 33 \quad \{3 \times (1)\}$

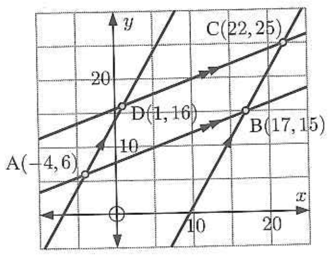
$-4r - 6t = -18 \quad \{2 \times (2)\}$

Adding, $5r = 15$

$\therefore \quad r = 3$

So, $\binom{x}{y} = \binom{-1}{6} + 3\binom{3}{-2} = \binom{8}{0}$ ✓

2 a



b A(−4, 6), B(17, 15), C(22, 25), D(1, 16)

c Lines 1 and 2 meet at A.

$\therefore \quad \binom{-4}{6} + r\binom{7}{3} = \binom{-4}{6} + s\binom{1}{2}$

$\therefore \quad \binom{7r-s}{3r-2s} = \binom{0}{0}$

$\therefore \quad 7r - s = 0 \quad .... (1)$

and $3r - 2s = 0$

$-14r + 2s = 0 \quad \{-2 \times (1)\}$

Adding, $-11r = 0$

$\therefore \quad r = 0$

$\therefore \quad \binom{x}{y} = \binom{-4}{6}$ ✓

Lines 1 and 4 meet at B.

$\therefore \quad \binom{-4}{6} + r\binom{7}{3} = \binom{22}{25} + u\binom{-1}{-2}$

$\therefore \quad \binom{7r+u}{3r+2u} = \binom{26}{19}$

$\therefore \quad 7r + u = 26 \quad .... (1)$

and $3r + 2u = 19$

$-14r - 2u = -52 \quad \{-2 \times (1)\}$

Adding, $-11r = -33$

$\therefore \quad r = 3$

$\therefore \quad \binom{x}{y} = \binom{-4}{6} + 3\binom{7}{3} = \binom{17}{15}$ ✓

Lines 3 and 4 meet at C.

$\therefore \quad \binom{22}{25} + t\binom{-7}{-3} = \binom{22}{25} + u\binom{-1}{-2}$

$\therefore \quad \binom{-7t+u}{-3t+2u} = \binom{0}{0}$

$\therefore \quad -7t + u = 0 \quad .... (1)$

and $-3t + 2u = 0$

$14t - 2u = 0 \quad \{-2 \times (1)\}$

Adding, $11t = 0$

$\therefore \quad t = 0$

$\therefore \quad \binom{x}{y} = \binom{22}{25}$ ✓

Lines 2 and 3 meet at D.

$\therefore \quad \binom{-4}{6} + s\binom{1}{2} = \binom{22}{25} + t\binom{-7}{-3}$

$\therefore \quad \binom{s+7t}{2s+3t} = \binom{26}{19}$

$\therefore \quad s + 7t = 26 \quad .... (1)$

and $2s + 3t = 19$

$-2s - 14t = -52 \quad \{-2 \times (1)\}$

Adding, $-11t = -33$

$\therefore \quad t = 3$

$\therefore \quad \binom{x}{y} = \binom{22}{25} + 3\binom{-7}{-3} = \binom{1}{16}$ ✓

3 a Lines 1 and 3 meet at A.

$\therefore \quad \binom{0}{2} + r\binom{2}{1} = \binom{0}{5} + t\binom{1}{-1}$

$\therefore \quad \binom{2r-t}{r+t} = \binom{0}{3}$

$\therefore \quad 2r - t = 0$

$r + t = 3$

Adding, $3r = 3$

$\therefore \quad r = 1$

$\therefore \quad \binom{x}{y} = \binom{0}{2} + \binom{2}{1} = \binom{2}{3}$

$\therefore$ A is (2, 3)

Lines 1 and 2 meet at B.

$\therefore \quad \binom{0}{2} + r\binom{2}{1} = \binom{8}{6} + s\binom{-1}{-2}$

$\therefore \quad \binom{2r+s}{r+2s} = \binom{8}{4}$

$\therefore \quad -4r - 2s = -16$

$r + 2s = 4$

Adding, $-3r = -12$

$\therefore \quad r = 4$

$\therefore \quad \binom{x}{y} = \binom{0}{2} + 4\binom{2}{1} = \binom{8}{6}$

$\therefore$ B is (8, 6)

Lines 2 and 3 meet at C.

$\therefore \quad \binom{8}{6} + s\binom{-1}{-2} = \binom{0}{5} + t\binom{1}{-1}$

$\therefore \quad \binom{-s-t}{-2s+t} = \binom{-8}{-1}$

$\therefore \quad -s - t = -8$

$-2s + t = -1$

Adding, $-3s = -9$

$\therefore \quad s = 3$

$\therefore \quad \binom{x}{y} = \binom{8}{6} + 3\binom{-1}{-2} = \binom{5}{0}$

$\therefore$ C is (5, 0)

b A(2, 3), B(8, 6), C(5, 0)

$\text{AB} = \sqrt{(8-2)^2 + (6-3)^2}$

$= \sqrt{36+9}$

$= \sqrt{45}$

$\text{BC} = \sqrt{(5-8)^2 + (0-6)^2}$

$= \sqrt{9+36}$

$= \sqrt{45}$

The two equal sides are [AB] and [BC] and they have length $\sqrt{45}$ units.

4 a Lines (QP) and (PR) meet at P.

$\therefore \quad \binom{3}{-1} + r\binom{14}{10} = \binom{0}{18} + t\binom{5}{-7}$

$\therefore \quad \binom{14r-5t}{10r+7t} = \binom{-3}{19}$

$\therefore \quad 14r - 5t = -3 \quad .... (1)$

$10r + 7t = 19 \quad .... (2)$

$\therefore \quad 98r - 35t = -21 \quad \{7 \times (1)\}$

$50r + 35t = 95 \quad \{5 \times (2)\}$

Adding, $148r = 74$

$\therefore \quad r = \frac{1}{2}$

$\therefore \quad \binom{x}{y} = \binom{3}{-1} + \frac{1}{2}\binom{14}{10} = \binom{10}{4}$

$\therefore$ P is (10, 4)

Lines (QP) and (PR) meet at Q.

$\binom{3}{-1} + r\binom{14}{10} = \binom{3}{-1} + s\binom{17}{-9}$

$\therefore \quad r\binom{14}{10} = s\binom{17}{-9}$

$\therefore \quad r = s = 0$

So, $\binom{x}{y} = \binom{3}{-1}$

$\therefore$ Q is (3, −1)

Lines (QR) and (PR) meet at R.

$\therefore \quad \binom{3}{-1} + s\binom{17}{-9} = \binom{0}{18} + t\binom{5}{-7}$

$\therefore \quad \binom{17s-5t}{-9s+7t} = \binom{-3}{19}$

$\therefore \quad 17s - 5t = -3 \quad .... (1)$

$-9s + 7t = 19 \quad .... (2)$

$\therefore \quad 119s - 35t = -21 \quad \{7 \times (1)\}$

$-45s + 35t = 95 \quad \{5 \times (2)\}$

Adding, $74s = 74$

$\therefore \quad s = 1$

$\therefore \quad \binom{x}{y} = \binom{3}{-1} + \binom{17}{-9} = \binom{20}{-10}$

$\therefore$ R is (20, −10)

b $\overrightarrow{PQ} = \binom{3-10}{-1-4} = \binom{-7}{-5}$

$\overrightarrow{PR} = \binom{20-10}{-10-4} = \binom{10}{-14}$

and $\overrightarrow{PQ} \bullet \overrightarrow{PR} = -70 + 70 = 0$

c $[PQ] \perp [PR] \quad \therefore \quad \widehat{QPR} = 90^\circ$

d Area $= \frac{1}{2}|\overrightarrow{PQ}||\overrightarrow{PR}|$

$= \frac{1}{2}\sqrt{49+25}\sqrt{100+196}$

$= 74 \text{ units}^2$

5 **a** Lines 1 and 4 meet at A.

$\therefore \begin{pmatrix}2\\5\end{pmatrix} + r\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} + u\begin{pmatrix}-3\\12\end{pmatrix}$

$\therefore \begin{pmatrix}4r+3u\\r-12u\end{pmatrix} = \begin{pmatrix}1\\-4\end{pmatrix}$

$\therefore \quad 4r + 3u = 1$
$\quad r - 12u = -4 \quad .... (1)$

$\therefore \quad 4r + 3u = 1$
$\quad -4r + 48u = 16 \quad \{-4 \times (1)\}$

Adding, $\quad 51u = 17$

$\therefore \quad u = \frac{1}{3}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-3\\12\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$

$\therefore$ A is (2, 5)

Lines 2 and 3 meet at C.

$\therefore \begin{pmatrix}18\\9\end{pmatrix} + s\begin{pmatrix}-8\\32\end{pmatrix} = \begin{pmatrix}14\\25\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix}$

$\therefore \begin{pmatrix}-8s+8t\\32s+2t\end{pmatrix} = \begin{pmatrix}-4\\16\end{pmatrix}$

$\therefore \quad -8s + 8t = -4 \quad .... (1)$
$\quad 32s + 2t = 16$

$\therefore \quad 2s - 2t = 1 \quad \{(1) \div -4\}$
$\quad 32s + 2t = 16$

Adding, $\quad 34s \quad = 17$

$\therefore \quad s = \frac{1}{2}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}18\\9\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-8\\32\end{pmatrix} = \begin{pmatrix}14\\25\end{pmatrix}$

$\therefore$ C is (14, 25)

Lines 1 and 2 meet at B.

$\therefore \begin{pmatrix}2\\5\end{pmatrix} + r\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}18\\9\end{pmatrix} + s\begin{pmatrix}-8\\32\end{pmatrix}$

$\therefore \begin{pmatrix}4r+8s\\r-32s\end{pmatrix} = \begin{pmatrix}16\\4\end{pmatrix}$

$\therefore \quad 4r + 8s = 16 \quad .... (1)$
$\quad r - 32s = 4 \quad .... (2)$

$\therefore \quad r + 2s = 4 \quad \{(1) \div 4\}$
$\quad -r + 32s = -4 \quad \{-1 \times (2)\}$

Adding, $\quad 34s = 0$

$\therefore \quad s = 0$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}18\\9\end{pmatrix}$

$\therefore$ B is (18, 9)

Lines 3 and 4 meet at D.

$\therefore \begin{pmatrix}14\\25\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} + u\begin{pmatrix}-3\\12\end{pmatrix}$

$\therefore \begin{pmatrix}-8t+3u\\-2t-12u\end{pmatrix} = \begin{pmatrix}-11\\-24\end{pmatrix}$

$\therefore \quad -8t + 3u = -11 \quad .... (1)$
$\quad -2t - 12u = -24 \quad .... (2)$

$\therefore \quad 16t - 6u = 22 \quad \{(-2) \times (1)\}$
$\quad t + 6u = 12 \quad \{(2) \div -2\}$

Adding, $\quad 17t \quad = 34$

$\therefore \quad t = 2$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}14\\25\end{pmatrix} + 2\begin{pmatrix}-8\\-2\end{pmatrix} = \begin{pmatrix}-2\\21\end{pmatrix}$

$\therefore$ D is (−2, 21)

b $\overrightarrow{AC} = \begin{pmatrix}14-2\\25-5\end{pmatrix} = \begin{pmatrix}12\\20\end{pmatrix}$

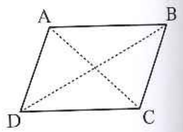
$\overrightarrow{DB} = \begin{pmatrix}18--2\\9-21\end{pmatrix} = \begin{pmatrix}20\\-12\end{pmatrix}$

i $|\overrightarrow{AC}| = \sqrt{12^2 + 20^2} = \sqrt{544}$ units

ii $|\overrightarrow{DB}| = \sqrt{20^2 + (-12)^2} = \sqrt{544}$ units

iii $\overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$

c The diagonals are perpendicular and equal in length, and as their midpoints are the same (at (8, 15)), ABCD is a square.



EXERCISE 13C

1 **a** Line 1 has direction vector $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $\quad 1 + 2t = -2 + 3s \qquad 2 - t = 3 - s \qquad 3 + t = 1 + 2s$

$\therefore \quad 2t - 3s = -3 \quad .... (1) \qquad \therefore \quad -t + s = 1 \quad .... (2) \qquad \therefore \quad t - 2s = -2 \quad .... (3)$

Solving (2) and (3) simultaneously: $\quad -t + s = 1$
$\quad t - 2s = -2$

$\quad -s = -1 \quad \therefore \quad s = 1$ and $t = 0$

and in (1), LHS $= 2t - 3s = 2(0) - 3(1) = -3 \quad \checkmark$

$\therefore \quad s = 1, \quad t = 0$ satisfies all three equations

$\therefore$ the two lines meet at (1, 2, 3) $\qquad$ {using $t = 0$ or $s = 1$}

The acute angle between the lines has $\quad \cos\theta = \dfrac{|6+1+2|}{\sqrt{4+1+1}\,\sqrt{9+1+4}} = \dfrac{9}{\sqrt{84}}$

and so $\quad \theta \approx 10.9°$

b Line 1 has direction vector $\begin{pmatrix}2\\-12\\12\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}4\\3\\-1\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $\quad -1 + 2t = 4s - 3 \qquad 2 - 12t = 3s + 2 \qquad 4 + 12t = -s - 1$

$\therefore \quad 2t - 4s = -2 \qquad -12t - 3s = 0 \qquad 12t + s = -5 \quad .... (3)$

$\therefore \quad t - 2s = -1 \quad .... (1) \qquad s = -4t \quad .... (2)$

Solving (1) and (2) simultaneously: $\quad t - 2(-4t) = -1$

$\therefore \quad 9t = -1$

$\therefore \quad t = -\frac{1}{9}$ and so $s = \frac{4}{9}$

In (3), $\quad 12t + s = 12\left(-\frac{1}{9}\right) + \frac{4}{9} = -\frac{12}{9} + \frac{4}{9} = -\frac{8}{9}$, which is not -5.

Since the system is inconsistent, the lines do not intersect, so the lines are skew.

The acute angle between the lines has $\quad \cos\theta = \dfrac{|8 - 36 - 12|}{\sqrt{292}\,\sqrt{26}} = \dfrac{40}{\sqrt{7592}}$ and so $\theta \approx 62.7°$.

c Line 1 has direction vector $\begin{pmatrix}6\\8\\2\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}3\\4\\1\end{pmatrix}$.

As $\begin{pmatrix}6\\8\\2\end{pmatrix} = 2\begin{pmatrix}3\\4\\1\end{pmatrix}$ the two lines are parallel. Hence, $\theta = 0°$.

d In line 1 set $\quad x = 2 - y = z + 2 = t, \quad$ so $\quad x = t, \quad y = 2 - t \quad$ and $\quad z = t - 2$.

Line 1 has direction vector $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}3\\-2\\2\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $\quad t = 1 + 3s \quad .... (1) \qquad 2 - t = -2 - 2s \qquad -2 + t = 2s + \frac{1}{2}$

$\qquad -t + 2s = -4 \quad .... (2) \qquad t - 2s = 2\frac{1}{2} \quad .... (3)$

Solving (1) and (2) simultaneously, $\quad -(1 + 3s) + 2s = -4$

$\therefore \quad -s = -3$

$\therefore \quad s = 3$ and so $t = 1 + 3(3) = 10$

Substituting in (3), $\quad t - 2s = 10 - 2(3) = 4 \neq 2\frac{1}{2}$

Since the system is inconsistent, the lines do not meet. $\therefore$ they are skew.

The acute angle between the lines has $\quad \cos\theta = \dfrac{|3+2+2|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \dfrac{7}{\sqrt{3}\sqrt{17}}$

$\therefore \quad \theta \approx 11.4°$

e Line 1 has direction vector $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}3\\-2\\1\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

$1 + t = 2 + 3s \qquad 2 - t = 3 - 2s \qquad 3 + 2t = s - 5$

$t - 3s = 1 \quad .... (1) \qquad -t + 2s = 1 \quad .... (2) \qquad 2t - s = -8 \quad .... (3)$

Solving (1) and (2) simultaneously, $\quad t - 3s = 1$
$\quad -t + 2s = 1$

Adding, $\quad -s = 2$

$\therefore \quad s = -2$ and $t - 3(-2) = 1 \quad \therefore \quad t = -5$