

c P_n is “ $A^{6n+5} = I - A$ ” for all n in \mathbb{Z}^+ .

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $A^{11} = A^5 A^6$
 $= (I - A)I$
 $= I - A \therefore P_1$ is true

(2) If P_k is true, then $A^{6k+5} = I - A$
 $\therefore A^{6(k+1)+5} = A^{6k+6+5}$
 $= A^{6k+5} A^6$
 $= (I - A)I$
 $= I - A$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

6 a $\begin{pmatrix} a & b & 0 \\ b & 0 & a \\ 0 & 2 & b \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2a+b & a+b & a+b \\ 2a+2b & 2a+b & a+b \\ 2+2b & 2+2b & 2+b \end{pmatrix}$

which is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ if $a+b=0$ and $2+2b=0$
 $\therefore b=-1$ and $a=1$.

The inverse is $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}$.

b Writing the system of equations in matrix form

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}$$

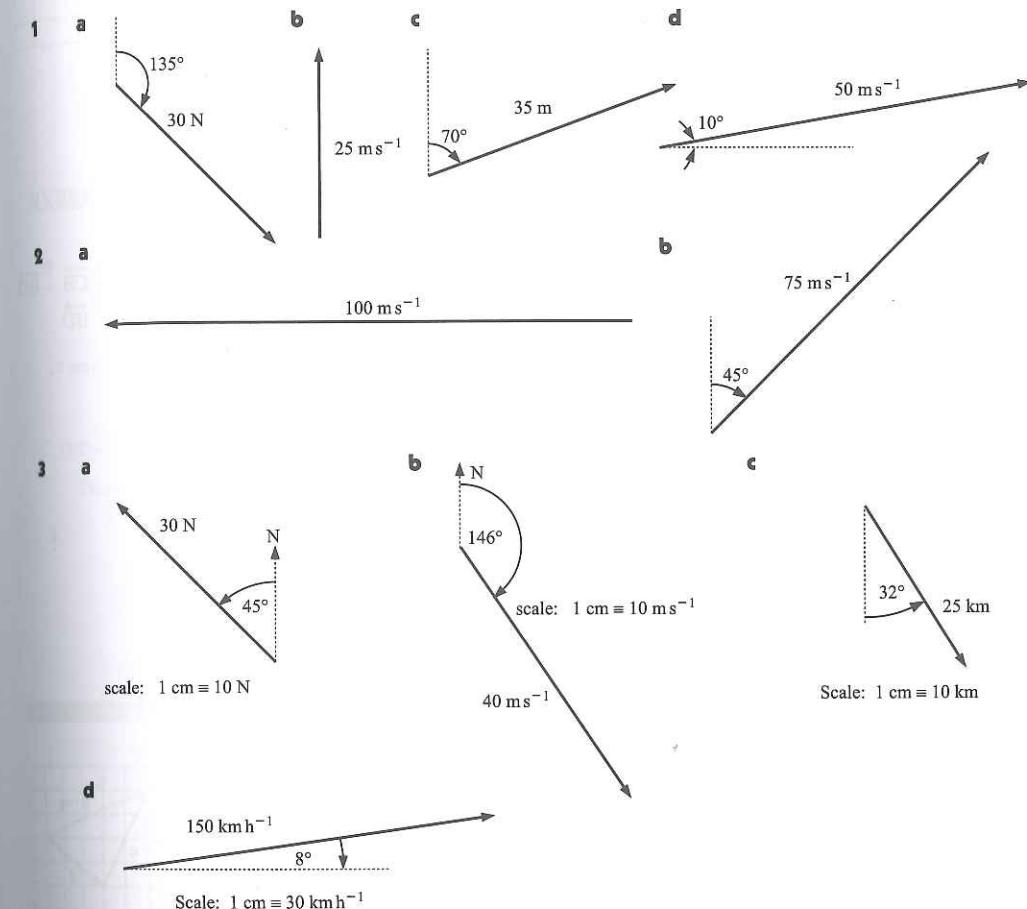
$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 7 \end{pmatrix}$$

$\therefore x = -5, y = 4, z = 7$.

Chapter 14

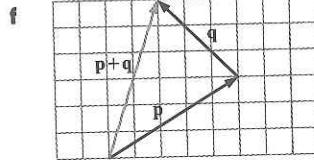
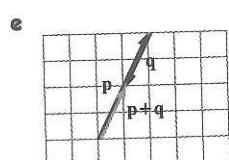
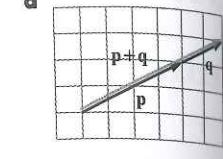
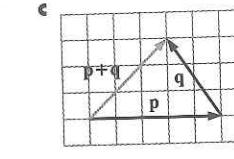
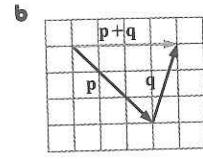
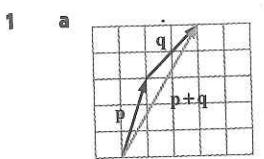
VECTORS IN 2 AND 3 DIMENSIONS

EXERCISE 14A.1



EXERCISE 14A.2

- 1 a If they are equal in magnitude, they have the same length. These are p, q, s and t .
 b Those parallel are p, q, r and t .
 c Those in the same direction are: p and r , q and t .
 d To be equal they must have the same direction and be equal in length $\therefore q = t$.
 e p and q are negatives (equal length, but opposite direction). Likewise, p and t are negatives. We write $p = -q$ and $p = -t$.
- 2 a True, as they have the same length and are parallel.
 b True, as they are sides of an equilateral triangle.
 c False, as they do not have the same direction.
 d False, as they have opposite directions.
 e True, as they have the same length and direction.
 f False, as they do not have the same direction.

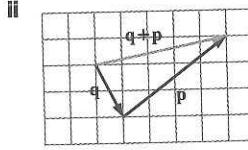
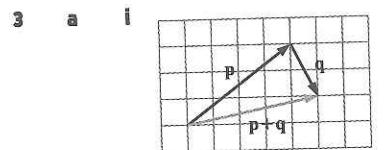
EXERCISE 14B.1

2 a $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

b $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

c $\begin{aligned} \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ = \overrightarrow{AC} + \overrightarrow{CD} \\ = \overrightarrow{AD} \end{aligned}$

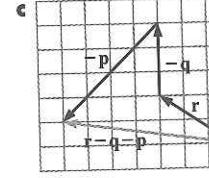
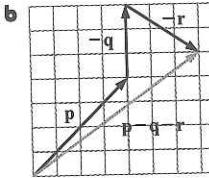
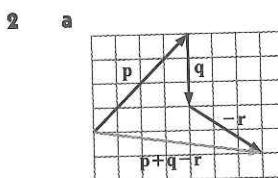
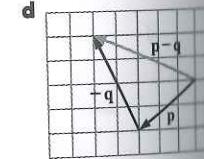
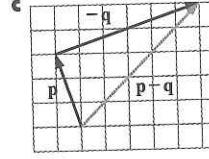
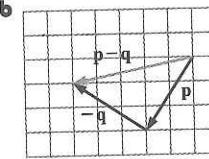
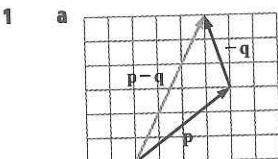
d $\begin{aligned} \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD} \\ = \overrightarrow{AB} + \overrightarrow{BD} \\ = \overrightarrow{AD} \end{aligned}$



b yes

4 $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$
 $= (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

But $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$
 $= \mathbf{a} + (\mathbf{b} + \mathbf{c})$
 $\therefore (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {as both are equal to \overrightarrow{PS} }

EXERCISE 14B.2

3 a $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

b $\begin{aligned} \overrightarrow{AD} - \overrightarrow{BD} \\ = \overrightarrow{AD} + \overrightarrow{DB} \\ = \overrightarrow{AB} \end{aligned}$

c $\begin{aligned} \overrightarrow{AC} + \overrightarrow{CA} \\ = \overrightarrow{AA} \\ = \mathbf{0} \end{aligned}$

d $\begin{aligned} \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ = \overrightarrow{AC} + \overrightarrow{CD} \\ = \overrightarrow{AD} \end{aligned}$

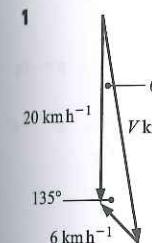
e $\begin{aligned} \overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB} \\ = \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CB} \\ = \overrightarrow{BC} + \overrightarrow{CB} \\ = \overrightarrow{BB} \\ = \mathbf{0} \end{aligned}$

f $\begin{aligned} \overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC} \\ = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ = \overrightarrow{AC} + \overrightarrow{CD} \\ = \overrightarrow{AD} \end{aligned}$

4 a $\mathbf{t} = \mathbf{r} + \mathbf{s}$
b $\mathbf{r} = -\mathbf{s} - \mathbf{t}$
e $\mathbf{p} = \mathbf{t} + \mathbf{s} + \mathbf{r} - \mathbf{q}$
f $\mathbf{p} = -\mathbf{u} + \mathbf{t} + \mathbf{s} - \mathbf{r} - \mathbf{q}$

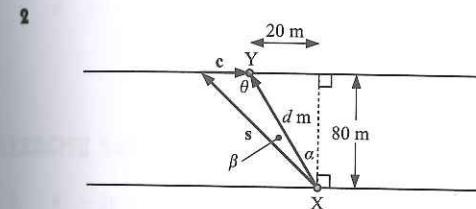
5 a i $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
ii $\begin{aligned} \overrightarrow{CA} &= \overrightarrow{CB} + \overrightarrow{BA} \\ &= -\overrightarrow{BC} - \overrightarrow{AB} \\ &= -\mathbf{t} - \mathbf{s} \end{aligned}$

b i $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$
ii $\begin{aligned} \overrightarrow{BC} &= \overrightarrow{BD} + \overrightarrow{DC} \\ &= \mathbf{p} + \mathbf{q} \end{aligned}$
iii $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC}$
= $\mathbf{p} + \mathbf{q} + \mathbf{r}$

EXERCISE 14B.3

a Using the cosine rule,
 $V^2 = 20^2 + 6^2 - 2 \times 20 \times 6 \times \cos 135^\circ$
 $\therefore V \approx 24.6$
 \therefore the equivalent speed in still water
is 24.6 km h^{-1} .

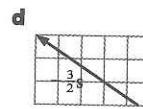
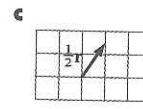
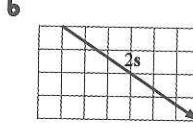
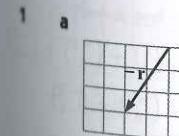
b Using the sine rule,
 $\frac{\sin \theta}{6} \approx \frac{\sin 135^\circ}{24.6}$
 $\therefore \theta \approx \sin^{-1} \left(\frac{6 \times \sin 135^\circ}{24.6} \right)$
 $\therefore \theta \approx 9.93^\circ$
 \therefore the boat should head
 9.93° east of south.

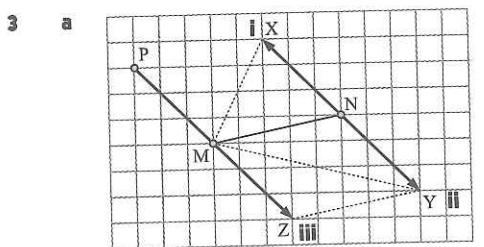
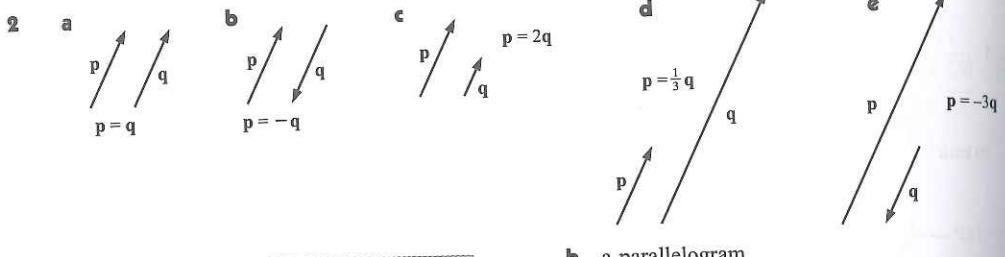
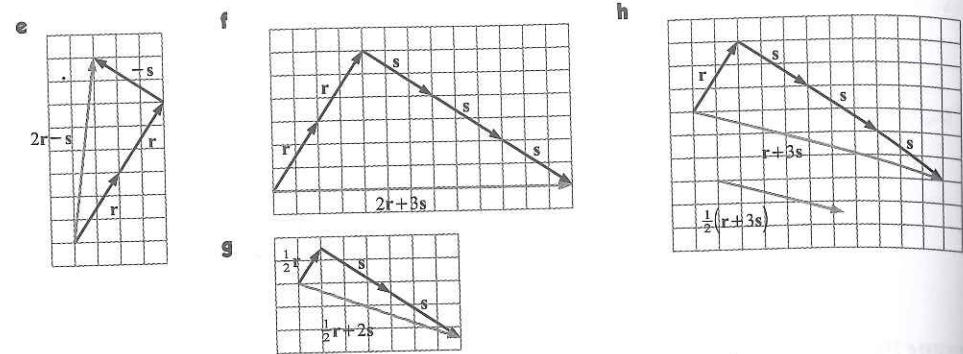
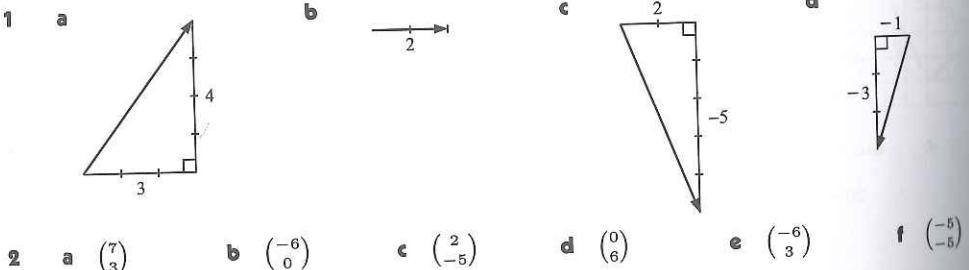


a $d^2 = 80^2 + 20^2$ {Pythagoras}
 $\therefore d = \sqrt{80^2 + 20^2}$ { $d > 0$ }
 $\therefore d \approx 82.5$
 \therefore the distance from X to Y is about 82.5 m.

c $\tan(\alpha + \beta) = \frac{20 + 0.3t}{80}$
 $\therefore 20 + 0.3t \approx 80 \tan(23.3^\circ)$
 $\therefore t \approx \frac{80 \tan(23.3^\circ) - 20}{0.3}$
 $\therefore t \approx 48.4$

\therefore Stephanie will take 48.4 seconds to cross the river.

EXERCISE 14B.4

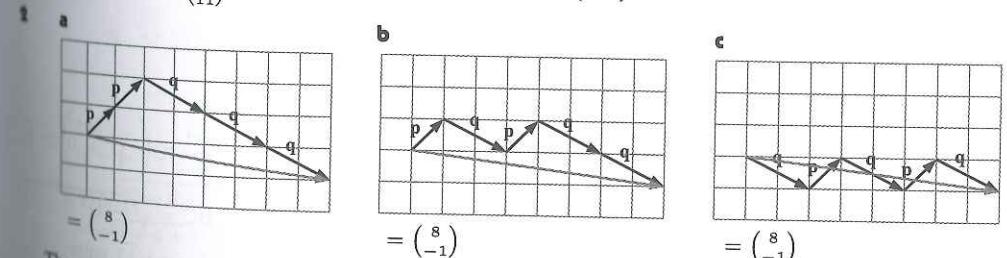
**EXERCISE 14C.1****EXERCISE 14C.2**

1	a $\mathbf{a} + \mathbf{b}$ $= \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$	b $\mathbf{b} + \mathbf{a}$ $= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$	c $\mathbf{b} + \mathbf{c}$ $= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ $= \begin{pmatrix} -1 \\ -1 \end{pmatrix}$	d $\mathbf{c} + \mathbf{b}$ $= \begin{pmatrix} -2 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} -1 \\ -1 \end{pmatrix}$
e	$\mathbf{a} + \mathbf{c}$ $= \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ $= \begin{pmatrix} -5 \\ -3 \end{pmatrix}$	f $\mathbf{c} + \mathbf{a}$ $= \begin{pmatrix} -2 \\ -5 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} -5 \\ -3 \end{pmatrix}$	g $\mathbf{a} + \mathbf{a}$ $= \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} -6 \\ 4 \end{pmatrix}$	h $\mathbf{b} + \mathbf{a} + \mathbf{c}$ $= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ $= \begin{pmatrix} -2 \\ -5 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ $= \begin{pmatrix} -4 \\ -10 \end{pmatrix}$

2	a $\mathbf{p} - \mathbf{q}$ $= \begin{pmatrix} -4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ $= \begin{pmatrix} -3 \\ 7 \end{pmatrix}$	b $\mathbf{q} - \mathbf{r}$ $= \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $= \begin{pmatrix} -4 \\ -3 \end{pmatrix}$	c $\mathbf{p} + \mathbf{q} - \mathbf{r}$ $= \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $= \begin{pmatrix} -8 \\ -1 \end{pmatrix}$
d	$\mathbf{p} - \mathbf{q} - \mathbf{r}$ $= \begin{pmatrix} -4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $= \begin{pmatrix} -6 \\ 9 \end{pmatrix}$	e $\mathbf{q} - \mathbf{r} - \mathbf{p}$ $= \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 0 \\ -5 \end{pmatrix}$	f $\mathbf{r} + \mathbf{q} - \mathbf{p}$ $= \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 6 \\ -9 \end{pmatrix}$
3	a \overrightarrow{AC} $= \overrightarrow{AB} + \overrightarrow{BC}$ $= -\overrightarrow{BA} + \overrightarrow{BC}$ $= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} -5 \\ 4 \end{pmatrix}$	b \overrightarrow{CB} $= \overrightarrow{CA} + \overrightarrow{AB}$ $= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$	c \overrightarrow{SP} $= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}$ $= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}$ $= -\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} 6 \\ -5 \end{pmatrix}$
4	a $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ $= \begin{pmatrix} 4 - 2 \\ 7 - 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$	b $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ $= \begin{pmatrix} 1 - 3 \\ 4 - 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$	c $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ $= \begin{pmatrix} 1 - 2 \\ 4 - 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$
d	$\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ $= \begin{pmatrix} 3 - 2 \\ 0 - 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$	e $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ $= \begin{pmatrix} 6 - 0 \\ -1 - 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$	f $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ $= \begin{pmatrix} 0 - 1 \\ 0 - 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

EXERCISE 14C.3

1	a $-3\mathbf{p} = -3\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \end{pmatrix}$	b $\frac{1}{2}\mathbf{q} = \frac{1}{2}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$	c $2\mathbf{p} + \mathbf{q} = 2\begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} 2 \\ 10 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} 0 \\ 14 \end{pmatrix}$
d	$\mathbf{p} - 2\mathbf{q} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -4 \\ 8 \end{pmatrix}$ $= \begin{pmatrix} 5 \\ -3 \end{pmatrix}$	e $\mathbf{p} - \frac{1}{2}\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$ $= \begin{pmatrix} \frac{5}{2} \\ \frac{11}{2} \end{pmatrix}$	f $2\mathbf{p} + 3\mathbf{r} = 2\begin{pmatrix} 1 \\ 5 \end{pmatrix} + 3\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} 2 \\ 10 \end{pmatrix} + \begin{pmatrix} -9 \\ 3 \end{pmatrix}$ $= \begin{pmatrix} -7 \\ 7 \end{pmatrix}$
g	$2\mathbf{q} - 3\mathbf{r} = 2\begin{pmatrix} -2 \\ 4 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} -4 \\ 8 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ $= \begin{pmatrix} 5 \\ 11 \end{pmatrix}$	h $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r} = \begin{pmatrix} 2 \\ 10 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ -\frac{1}{3} \end{pmatrix}$ $= \begin{pmatrix} 3 \\ \frac{17}{3} \end{pmatrix}$	



The vector expressions are equal, as each consists of $2\mathbf{ps}$ and $3\mathbf{qs}$. Each expression is equal to $2\mathbf{p} + 3\mathbf{q}$.

EXERCISE 14C.4

1 a $|\mathbf{r}| = \sqrt{2^2 + 3^2} = \sqrt{13}$ units

b $|\mathbf{s}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$ units

2 a $|\mathbf{p}| = \sqrt{1^2 + 3^2} = \sqrt{10}$ units

d $3\mathbf{p} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$
 $\therefore |3\mathbf{p}| = \sqrt{3^2 + 9^2} = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10}$ units

g $4\mathbf{q} = \begin{pmatrix} -8 \\ 16 \end{pmatrix}$
 $\therefore |4\mathbf{q}| = \sqrt{(-8)^2 + 16^2} = \sqrt{64 + 256} = \sqrt{320} = 8\sqrt{5}$ units

i $\frac{1}{2}\mathbf{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
 $\therefore \left| \frac{1}{2}\mathbf{q} \right| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$ units

3 $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix} \therefore |k\mathbf{a}| = \sqrt{(ka_1)^2 + (ka_2)^2} = \sqrt{k^2a_1^2 + k^2a_2^2} = \sqrt{k^2(a_1^2 + a_2^2)} = \sqrt{k^2}\sqrt{a_1^2 + a_2^2} = |k|\sqrt{a_1^2 + a_2^2} = |k|\|\mathbf{a}\|$

4 a $\overrightarrow{AB} = \begin{pmatrix} 3-2 \\ 5-(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \therefore AB = \sqrt{1^2 + 6^2} = \sqrt{37}$ units

b $\overrightarrow{BA} = \begin{pmatrix} 2-3 \\ -1-5 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \end{pmatrix} \therefore BA = \sqrt{(-1)^2 + (-6)^2} = \sqrt{37}$ units

EXERCISE 14C.4

c $\mathbf{r} + \mathbf{s} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

d $\mathbf{r} - \mathbf{s} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

e $\mathbf{s} - 2\mathbf{r} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$

f $|\mathbf{r} + \mathbf{s}| = \sqrt{1^2 + 7^2} = \sqrt{50}$ units

g $|\mathbf{r} - \mathbf{s}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

h $|\mathbf{s} - 2\mathbf{r}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$ units

i $2\mathbf{p} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

j $-2\mathbf{p} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$
 $\therefore |2\mathbf{p}| = \sqrt{2^2 + 6^2} = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}$ units

k $-3\mathbf{p} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$
 $\therefore |-3\mathbf{p}| = \sqrt{(-3)^2 + (-9)^2} = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10}$ units

l $4\mathbf{q} = \begin{pmatrix} -8 \\ 16 \end{pmatrix}$
 $\therefore |4\mathbf{q}| = \sqrt{(-8)^2 + 16^2} = \sqrt{64 + 256} = \sqrt{320} = 8\sqrt{5}$ units

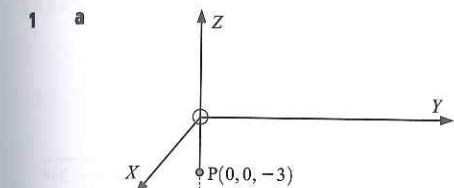
m $-\frac{1}{2}\mathbf{q} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
 $\therefore \left| -\frac{1}{2}\mathbf{q} \right| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$ units

d $\overrightarrow{DC} = \begin{pmatrix} -1 & -4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \quad e \quad \overrightarrow{CA} = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad f \quad \overrightarrow{DA} = \begin{pmatrix} 2 & -4 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

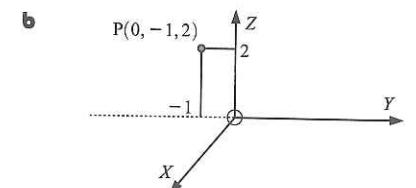
$\therefore DC = \sqrt{3^2 + 7^2} = \sqrt{58}$ units

$\therefore CA = \sqrt{2^2 + (-5)^2} = \sqrt{34}$ units

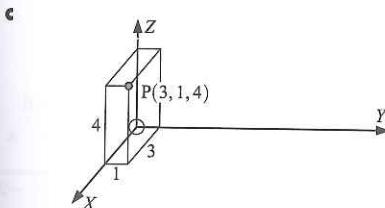
$\therefore DA = \sqrt{6^2 + 2^2} = 2\sqrt{10}$ units

EXERCISE 14D

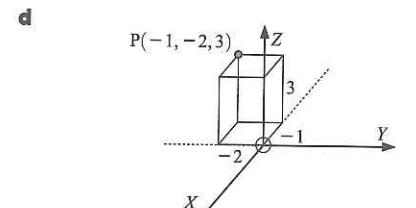
$OP = \sqrt{0^2 + 0^2 + (-3)^2} = 3$ units



$OP = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$ units



$OP = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$ units



$OP = \sqrt{(-1)^2 + (-2)^2 + 3^2} = \sqrt{14}$ units

2 a i $AB = \sqrt{(0-(-1))^2 + (-1-2)^2 + (1-3)^2} = \sqrt{1+9+4} = \sqrt{14}$ units

ii Midpoint is at $\left(\frac{-1+0}{2}, \frac{2-1}{2}, \frac{3+1}{2} \right)$ which is $(-\frac{1}{2}, \frac{1}{2}, 2)$.

c i $AB = \sqrt{(-1-3)^2 + (0-(-1))^2 + (1-(-1))^2} = \sqrt{16+1+4} = \sqrt{21}$ units

ii Midpoint is at $\left(\frac{3-1}{2}, \frac{-1+0}{2}, \frac{-1+1}{2} \right)$ which is $(1, -\frac{1}{2}, 0)$.

3 P(0, 4, 4), Q(2, 6, 5), R(1, 4, 3)
 $PQ = \sqrt{(2-0)^2 + (6-4)^2 + (5-4)^2} = \sqrt{4+4+1} = 3$

$PR = \sqrt{(1-0)^2 + (4-4)^2 + (3-4)^2} = \sqrt{1+0+1} = \sqrt{2}$

d i $AB = \sqrt{(0-2)^2 + (1-0)^2 + (0-(-3))^2} = \sqrt{4+1+9} = \sqrt{14}$ units

ii Midpoint is at $\left(\frac{2+0}{2}, \frac{0+1}{2}, \frac{-3+0}{2} \right)$ which is $(1, \frac{1}{2}, -\frac{3}{2})$.

$$\begin{aligned} QR &= \sqrt{(1-2)^2 + (4-6)^2 + (3-5)^2} \\ &= \sqrt{1+4+4} \\ &= 3. \end{aligned}$$

$\therefore PQ = QR$ and so $\triangle PQR$ is isosceles.

- 4 a A(2, -1, 7), B(3, 1, 4), C(5, 4, 5)

$$\begin{aligned} AB &= \sqrt{(3-2)^2 + (1-(-1))^2 + (4-7)^2} \\ &= \sqrt{1+4+9} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(5-3)^2 + (4-1)^2 + (5-4)^2} \\ &= \sqrt{4+9+1} \\ &= \sqrt{14} \end{aligned}$$

Since $AB = BC$, $\triangle ABC$ is isosceles.

- b A(0, 0, 3) B(2, 8, 1) C(-9, 6, 18)

$$\begin{aligned} AB &= \sqrt{(2-0)^2 + (8-0)^2 + (1-3)^2} \\ &= \sqrt{4+64+4} \\ &= \sqrt{72} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-9-2)^2 + (6-8)^2 + (18-1)^2} \\ &= \sqrt{121+4+289} \\ &= \sqrt{414} \end{aligned}$$

Since $BC^2 = AB^2 + AC^2$, $\triangle ABC$ is right angled.

- c A(5, 6, -2) B(6, 12, 9) C(2, 4, 2)

$$\begin{aligned} AB &= \sqrt{(6-5)^2 + (12-6)^2 + (9-(-2))^2} \\ &= \sqrt{1+36+121} \\ &= \sqrt{158} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(2-6)^2 + (4-12)^2 + (2-9)^2} \\ &= \sqrt{16+64+49} \\ &= \sqrt{129} \end{aligned}$$

Since $AB^2 = AC^2 + BC^2$, $\triangle ABC$ is right angled.

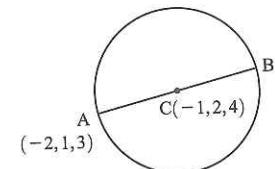
- d A(1, 0, -3) B(2, 2, 0) C(4, 6, 6)

$$\begin{aligned} AB &= \sqrt{(2-1)^2 + (2-0)^2 + (0-(-3))^2} \\ &= \sqrt{1^2+2^2+3^2} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(4-2)^2 + (6-2)^2 + (6-0)^2} \\ &= \sqrt{2^2+4^2+6^2} \\ &= \sqrt{56} \\ &= 2\sqrt{14} \end{aligned}$$

Since $AB + BC = AC$, the points A, B and C lie on a straight line, so they do not form a triangle.

5



$$\text{If } B \text{ is } (a, b, c) \text{ then } \frac{a-2}{2} = -1, \frac{b+1}{2} = 2, \frac{c+3}{2} = 4$$

$$\therefore a = 0, b = 3, c = 5$$

$$\therefore B \text{ is } (0, 3, 5)$$

$$\begin{aligned} r &= AC = \sqrt{(-1-(-2))^2 + (2-1)^2 + (4-3)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

- 6 a (0, y , 0) for any y

b The distance between (0, y , 0) and B(-1, -1, 2) is $\sqrt{(-1)^2 + (-1-y)^2 + 2^2}$.

$$\therefore \sqrt{1+(y+1)^2+4} = \sqrt{14}$$

$$\therefore (y+1)^2 = 9$$

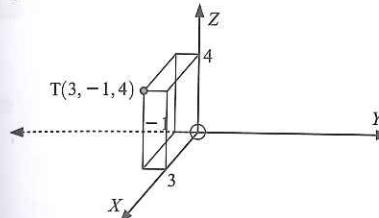
$$\therefore y+1 = \pm 3$$

$$\therefore y = -1 \pm 3$$

$$\therefore y = -4 \text{ or } 2 \quad \therefore \text{the two points are } (0, -4, 0) \text{ and } (0, 2, 0).$$

EXERCISE 14E.1

1 a



$$\overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \text{c } OT &= \sqrt{(3-0)^2 + (-1-0)^2 + (4-0)^2} \\ &= \sqrt{9+1+16} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

2 a

$$\begin{aligned} \overrightarrow{AB} &= \begin{pmatrix} 1-(-3) \\ 0-1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \\ \overrightarrow{BA} &= \begin{pmatrix} -3-1 \\ 1-0 \\ 2-(-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{b } |\overrightarrow{AB}| &= \sqrt{4^2 + (-1)^2 + (-3)^2} \\ &= \sqrt{26} \text{ units} \\ |\overrightarrow{BA}| &= \sqrt{(-4)^2 + 1^2 + 3^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

3

$$\begin{aligned} \overrightarrow{OA} &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} & \overrightarrow{OB} &= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} & \overrightarrow{AB} &= \begin{pmatrix} -1-3 \\ 1-1 \\ 2-0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

4

- a The position vector of M from N

$$= \overrightarrow{NM} = \begin{pmatrix} 4-(-1) \\ -2-2 \\ -1-0 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}.$$

$$\text{c } |\overrightarrow{MN}| = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{25+16+1} = \sqrt{42} \text{ units}$$

5

- a The position vector of A from O

$$= \overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}.$$

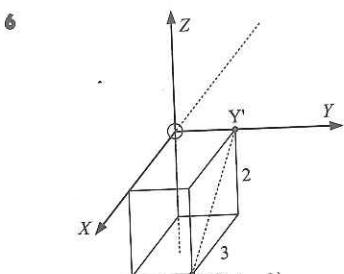
$$\therefore |\overrightarrow{OA}| = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{1+4+25} = \sqrt{30} \text{ units}$$

- c The position vector of B from C

$$= \overrightarrow{CB} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \text{b } \text{The position vector of N from M} \\ = \overrightarrow{MN} &= \begin{pmatrix} -1-4 \\ 2-(-2) \\ 0-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \text{b } \text{The position vector of C from A} \\ = \overrightarrow{AC} &= \begin{pmatrix} -3-(-1) \\ 1-2 \\ 0-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -5 \end{pmatrix}. \\ \therefore |\overrightarrow{AC}| &= \sqrt{(-2)^2 + (-1)^2 + (-5)^2} \\ &= \sqrt{4+1+25} \\ &= \sqrt{30} \text{ units} \\ \text{and } |\overrightarrow{CB}| &= \sqrt{5^2 + (-1)^2 + 3^2} \\ &= \sqrt{25+1+9} \\ &= \sqrt{35} \text{ units} \end{aligned}$$



- 6**
- a The distance from Q to the Y -axis is the distance from Q to $Y'(0, 1, 0)$
 $\therefore QY' = \sqrt{(3-0)^2 + (1-1)^2 + (-2-0)^2}$
 $= \sqrt{9+4}$
 $= \sqrt{13}$ units
- b The distance from Q to the origin is
 $QO = \sqrt{(3-0)^2 + (1-0)^2 + (-2-0)^2}$
 $= \sqrt{9+1+4}$
 $= \sqrt{14}$ units
- c The distance from Q to the ZOY plane is the distance from Q to $(0, 1, -2)$, which is 3 units.

EXERCISE 14E.2

1 a $\begin{pmatrix} a-4 \\ b-3 \\ c+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$

$$\therefore \begin{cases} a-4=1 \\ b-3=3 \\ c+2=-4 \end{cases}$$

$$\therefore a=5, b=6, c=-6$$

b $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$

$$\therefore \begin{cases} a-5=3-a \\ b-2=2-b \\ c+3=5-c \end{cases}$$

$$\therefore 2a=8, 2b=4, 2c=2$$

$$\therefore a=4, b=2, c=1$$

2 a $2 \begin{pmatrix} 1 \\ 0 \\ 3a \end{pmatrix} = \begin{pmatrix} b \\ c-1 \\ 2 \end{pmatrix}$

$$\therefore \begin{pmatrix} 2 \\ 0 \\ 6a \end{pmatrix} = \begin{pmatrix} b \\ c-1 \\ 2 \end{pmatrix}$$

$$\therefore 6a=2, b=2, c-1=0$$

$$\therefore a=\frac{1}{3}, b=2, c=1$$

b $\begin{pmatrix} 2 \\ a \\ 3 \end{pmatrix} = \begin{pmatrix} b \\ a^2 \\ a+b \end{pmatrix}$

$$\therefore b=2, a^2=a, a+b=3$$

$$\therefore a=1, b=2$$

c $a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

$$\therefore a+2b=-1 \dots (1), a+c=3 \dots (2) \text{ and } -b+c=3 \dots (3)$$

$$(1)-(2) \text{ gives: } 2b-c=-4 \dots (4)$$

Adding (3) and (4) gives $b=-1$

using (3), $c=2$

and using (2), $a=1$

$\therefore a=1, b=-1, c=2$

3 A(-1, 3, 4), B(2, 5, -1), C(-1, 2, -2), D(r, s, t)

a If $\overrightarrow{AC} = \overrightarrow{BD}$ then $\begin{pmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t+1 \end{pmatrix}$
 $\therefore r-2=0, s-5=-1 \text{ and } t+1=-6 \quad \therefore r=2, s=4 \text{ and } t=-7$

b If $\overrightarrow{AB} = \overrightarrow{DC}$ then $\begin{pmatrix} 2-(-1) \\ 5-3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1-r \\ 2-s \\ -2-t \end{pmatrix}$
 $\therefore -1-r=3, 2-s=2 \text{ and } -2-t=-5 \quad \therefore r=-4, s=0 \text{ and } t=3$

4 a $\overrightarrow{AB} = \begin{pmatrix} 3-1 \\ -3-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{DC} = \begin{pmatrix} 7-5 \\ -4-1 \\ 5-6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

5 a Suppose S is at (x, y, z) . $\overrightarrow{PQ} = \overrightarrow{SR}$ {opposite sides are parallel and equal in length}

$$\begin{array}{ccc} P(-1, 2, 3) & \xrightarrow{\hspace{1cm}} & Q(1, -2, 5) \\ & \searrow & \downarrow \\ S(x, y, z) & \xrightarrow{\hspace{1cm}} & R(0, 4, -1) \end{array}$$

$$\therefore \begin{pmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 0-x \\ 4-y \\ -1-z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -x \\ 4-y \\ -1-z \end{pmatrix}$$

$$\therefore -x=2 \quad 4-y=-4 \quad -1-z=2$$

$$\therefore x=-2 \quad y=8 \quad z=-3$$

$$\therefore S \text{ is at } (-2, 8, -3).$$

b The midpoint of [PR] is $\left(\frac{-1+0}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

The midpoint of [QS] is $\left(\frac{1+(-2)}{2}, \frac{-2+8}{2}, \frac{5+(-3)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

So, [PR] and [QS] have the same midpoint. ✓

EXERCISE 14F.1

1 a $2\mathbf{x} = \mathbf{q}$

$$\therefore \frac{1}{2}(2\mathbf{x}) = \frac{1}{2}\mathbf{q}$$

$$\therefore \mathbf{x} = \frac{1}{2}\mathbf{q}$$

b $\frac{1}{2}\mathbf{x} = \mathbf{n}$

$$\therefore 2(\frac{1}{2}\mathbf{x}) = 2\mathbf{n}$$

$$\therefore \mathbf{x} = 2\mathbf{n}$$

c $-3\mathbf{x} = \mathbf{p}$

$$\therefore 3\mathbf{x} = -\mathbf{p}$$

$$\therefore \frac{1}{3}(3\mathbf{x}) = -\frac{1}{3}\mathbf{p}$$

$$\therefore \mathbf{x} = -\frac{1}{3}\mathbf{p}$$

d $\mathbf{q} + 2\mathbf{x} = \mathbf{r}$

$$\therefore 2\mathbf{x} = \mathbf{r} - \mathbf{q}$$

$$\therefore \mathbf{x} = \frac{1}{2}(\mathbf{r} - \mathbf{q})$$

e $4\mathbf{s} - 5\mathbf{x} = \mathbf{t}$

$$\therefore -5\mathbf{x} = \mathbf{t} - 4\mathbf{s}$$

$$\therefore 5\mathbf{x} = 4\mathbf{s} - \mathbf{t}$$

f $4\mathbf{m} - \frac{1}{3}\mathbf{x} = \mathbf{n}$

$$\therefore 4\mathbf{m} - \mathbf{n} = \frac{1}{3}\mathbf{x}$$

$$\therefore \mathbf{x} = 12\mathbf{m} - 3\mathbf{n}$$

$$= 3(4\mathbf{m} - \mathbf{n})$$

2 a $2\mathbf{y} = \mathbf{r}$

$$\therefore \mathbf{y} = \frac{1}{2}\mathbf{r}$$

b $\frac{1}{2}\mathbf{y} = \mathbf{s}$

$$\therefore \mathbf{y} = 2\mathbf{s}$$

c $\mathbf{r} + 2\mathbf{y} = \mathbf{s}$

$$\therefore 2\mathbf{y} = \mathbf{s} - \mathbf{r}$$

$$\therefore \mathbf{y} = \frac{1}{2}\mathbf{s} - \frac{1}{2}\mathbf{r}$$

$$= \frac{1}{2} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{4} \\ \frac{3}{4} \end{pmatrix}$$

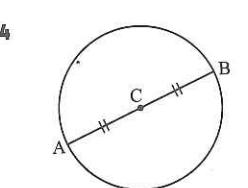
3 $k\mathbf{x} = \mathbf{a}$

$$\therefore k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\therefore kx_1 = a_1 \text{ and } kx_2 = a_2$$

$$\therefore x_1 = \frac{1}{k}a_1 \text{ and } x_2 = \frac{1}{k}a_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and so } \mathbf{x} = \frac{1}{k}\mathbf{a}$$



- 4 a $\vec{AC} = \begin{pmatrix} 1-3 \\ 4-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} \therefore B \text{ is } (1-2, 4+6) \text{ or } (-1, 10).$
 b $\vec{AC} = \begin{pmatrix} -1-0 \\ -2-5 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix} \therefore B \text{ is } (-1-1, -2-7) \text{ or } (-2, -9).$
 c $\vec{AC} = \begin{pmatrix} 3-1 \\ 0-4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \therefore B \text{ is } (3+4, 0+4) \text{ or } (7, 4).$

5 a M is $\left(\frac{3+1}{2}, \frac{6+2}{2}\right)$
 $\therefore M \text{ is } (1, 4)$
 b $\vec{CA} = \begin{pmatrix} 3-4 \\ 6-1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$
 $\vec{CM} = \begin{pmatrix} 1-4 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$
 $\vec{CB} = \begin{pmatrix} -1-4 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ which is } \vec{CM}$
 c $\frac{1}{2}\vec{CA} + \frac{1}{2}\vec{CB} = \frac{1}{2}\begin{pmatrix} 7 \\ 5 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

6 a $2\mathbf{a} + \mathbf{x} = \mathbf{b}$
 $\therefore \mathbf{x} = \mathbf{b} - 2\mathbf{a}$

$$\begin{aligned} &= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -6 \\ -5 \end{pmatrix} \end{aligned}$$

c $2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$
 $\therefore \mathbf{a} + 2\mathbf{b} = 2\mathbf{x}$

$$\begin{aligned} \therefore \mathbf{x} &= \frac{1}{2}(\mathbf{a} + 2\mathbf{b}) = \frac{1}{2} \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \quad \{\text{using b}\} \\ &= \begin{pmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{pmatrix} \end{aligned}$$

7 $\vec{AB} = \vec{AO} + \vec{OB} = -\vec{OA} + \vec{OB} = -\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$
 $\therefore |\vec{AB}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9 + 16 + 4} = \sqrt{29} \text{ units}$

8 $\vec{OA} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \vec{OB} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}, \quad \vec{OC} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{OD} = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$
 $\therefore \vec{BD} = \vec{BO} + \vec{OD} = -\vec{OB} + \vec{OD} = -\begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix}$
 and $\vec{AC} = \vec{AO} + \vec{OC} = -\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} \quad \therefore \vec{BD} = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 2\vec{AC}$

9 $\vec{AB} = \begin{pmatrix} 2-1 \\ 3-5 \\ -3-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} \therefore C \text{ is } (2+3, 3-2, -3-5), \text{ or } (5, 1, -8),$
 D is $(5+3, 1-2, -8-5), \text{ or } (8, -1, -13),$
 E is $(8+3, -1-2, -13-5), \text{ or } (11, -3, -18).$

10
 a $\vec{AB} = \begin{pmatrix} 4-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
 $\vec{DC} = \begin{pmatrix} -1-2 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Now $\vec{AB} = \vec{DC}$
 \therefore sides [AB] and [DC] are equal in length and parallel.

This is sufficient to deduce that ABCD is a parallelogram.

b $\vec{AB} = \begin{pmatrix} -1-5 \\ 2-0 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$
 $\vec{DC} = \begin{pmatrix} 4-10 \\ -3-5 \\ 6-5 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$

c $\vec{AB} = \begin{pmatrix} 1-2 \\ 4-3 \\ -1-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$
 $\vec{DC} = \begin{pmatrix} -2-1 \\ 6-1 \\ -2-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -4 \end{pmatrix}$

11 a Let D be (a, b) .
 Now $\vec{CD} = \vec{BA}$
 $\therefore \begin{pmatrix} a-8 \\ b-2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 0-1 \end{pmatrix} \therefore \begin{pmatrix} a-4 \\ b-0 \end{pmatrix} = \begin{pmatrix} -2-1 \\ 5-4 \end{pmatrix} \therefore \begin{pmatrix} a-1 \\ b-5 \end{pmatrix} = \begin{pmatrix} 3-0 \\ -2-4 \end{pmatrix}$
 $\therefore \begin{pmatrix} a-8 \\ b+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \therefore \begin{pmatrix} a-4 \\ b \\ c-7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \therefore \begin{pmatrix} a+1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}$
 So, D is $(9, -1)$.
 b Let R be (a, b, c) .
 Now $\vec{SR} = \vec{PQ}$
 $\therefore \begin{pmatrix} a-8 \\ b-2 \\ c-7 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 0-1 \\ 2-3 \end{pmatrix} \therefore \begin{pmatrix} a-4 \\ b \\ c-7 \end{pmatrix} = \begin{pmatrix} -2-1 \\ 5-4 \\ 2-3 \end{pmatrix} \therefore \begin{pmatrix} a+1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}$
 $\therefore a = 9, b = -1 \quad \therefore a = 3, b = 1, c = 6 \quad \therefore a = 2, b = -1, c = 0$
 So, R is $(3, 1, 6)$.
 c Let X be (a, b, c) .
 Now $\vec{WX} = \vec{ZY}$
 $\therefore \begin{pmatrix} a-8 \\ b-2 \\ c-7 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 0-1 \\ 2-3 \end{pmatrix} \therefore \begin{pmatrix} a-4 \\ b \\ c-7 \end{pmatrix} = \begin{pmatrix} -2-1 \\ 5-4 \\ 2-3 \end{pmatrix} \therefore \begin{pmatrix} a+1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}$
 $\therefore a = 9, b = -1 \quad \therefore a = 3, b = 1, c = 6 \quad \therefore a = 2, b = -1, c = 0$
 So, X is $(2, -1, 0)$.

12 a $\vec{BD} = \frac{1}{2}\vec{OA}$
 $= \frac{1}{2}\mathbf{a}$
 b $\vec{AB} = \vec{AO} + \vec{OB}$
 $= -\mathbf{a} + \mathbf{b}$
 $= \mathbf{b} - \mathbf{a}$
 d $\vec{OD} = \vec{OB} + \vec{BD}$
 $= \mathbf{b} + \frac{1}{2}\mathbf{a}$
 e $\vec{AD} = \vec{AO} + \vec{OD}$
 $= -\mathbf{a} + \mathbf{b} + \frac{1}{2}\mathbf{a}$
 $= -\frac{1}{2}\mathbf{a} + \mathbf{b}$
 f $\vec{DA} = -\vec{AD}$
 $= \frac{1}{2}\mathbf{a} - \mathbf{b}$

13 a $\vec{AD} = \vec{AB} + \vec{BD} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$
 b $\vec{CB} = \vec{CA} + \vec{AB} = -\vec{AC} + \vec{AB} = -\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}$
 c $\vec{CD} = \vec{CB} + \vec{BD} = \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \quad \{\text{using b}\} = \begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$

14 a $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ b $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

c $\mathbf{b} + 2\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$

d $\mathbf{a} - 3\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 10 \end{pmatrix}$

e $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$

f $\mathbf{c} - \frac{1}{2}\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{7}{2} \end{pmatrix}$

g $\mathbf{a} - \mathbf{b} - \mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$

h $2\mathbf{b} - \mathbf{c} + \mathbf{a} = 2 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

15 a $|\mathbf{a}| = \sqrt{(-1)^2 + 1^2 + 3^2} = \sqrt{11}$ units

b $|\mathbf{b}| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$ units

c $|\mathbf{b} + \mathbf{c}| = \left| \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \right| = \left| \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix} \right| = \sqrt{(-1)^2 + (-1)^2 + 6^2} = \sqrt{1 + 1 + 36} = \sqrt{38}$ units

e $|\mathbf{a}| \mathbf{b} = \sqrt{11} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{pmatrix}$

f $\frac{1}{|\mathbf{a}|} \mathbf{a} = \frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix}$

16 a $r \begin{pmatrix} 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -8 \\ -27 \end{pmatrix}$

$\therefore \begin{pmatrix} r+2s \\ -r+5s \end{pmatrix} = \begin{pmatrix} -8 \\ -27 \end{pmatrix}$

$\therefore r+2s = -8 \quad \dots \text{(1)}$

$-r+5s = -27$

adding $7s = -35$

$\therefore s = -5$

and in (1) $r+2(-5) = -8$

$\therefore r-10 = -8$

$\therefore r = 2$

So, $r = 2, s = -5$

b $r \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -19 \\ 2 \end{pmatrix}$

$\therefore \begin{cases} 2r+s=7 & \dots \text{(1)} \\ -3r+7s=-19 & \dots \text{(2)} \\ r+2s=2 & \dots \text{(3)} \end{cases}$

$\therefore -4r-2s=-14 \quad \{-2 \times \text{(1)}\}$

$r+2s=2$

adding $-3r = -12$

$\therefore r=4$

In (1), $2(4)+s=7 \quad \therefore s=-1$

Checking in (2),

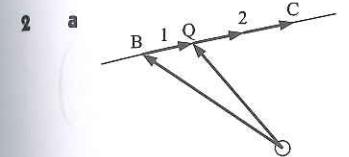
$-3r+7s=-3(4)+7(-1)=-19$

$\therefore r=4, s=-1$ satisfies all equations.

EXERCISE 14F.2

1 a $\overrightarrow{AB} : \overrightarrow{BC} = 1 : 1$
 $\therefore B$ divides $[AC]$ in the ratio $1 : 1$.

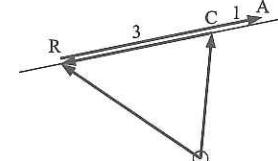
d $\overrightarrow{ED} : \overrightarrow{DA} = 1 : 3$
 $\therefore D$ divides $[EA]$ in the ratio $1 : 3$.



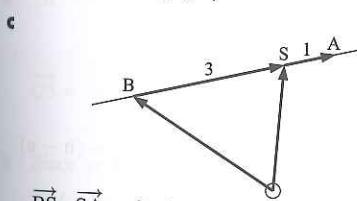
$$\begin{aligned} \overrightarrow{BQ} : \overrightarrow{QC} &= 1 : 2 \\ \therefore \overrightarrow{BQ} &= \frac{1}{3} \overrightarrow{BC} \\ \overrightarrow{OQ} &= \overrightarrow{OB} + \overrightarrow{BQ} \\ &= \overrightarrow{OB} + \frac{1}{3} \overrightarrow{BC} \\ &= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 - (-1) \\ -1 - 2 \\ 4 - 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} \\ 1 \\ \frac{4}{3} \end{pmatrix} \quad \therefore Q \text{ is } \left(-\frac{1}{3}, 1, \frac{4}{3}\right). \end{aligned}$$

b $\overrightarrow{BC} : \overrightarrow{CF} = 1 : 3$
 $\therefore C$ divides $[BF]$ in the ratio $1 : 3$.

e $\overrightarrow{AC} : \overrightarrow{CB} = -2 : 1$
 $\therefore C$ divides $[AB]$ in the ratio $-2 : 1$.



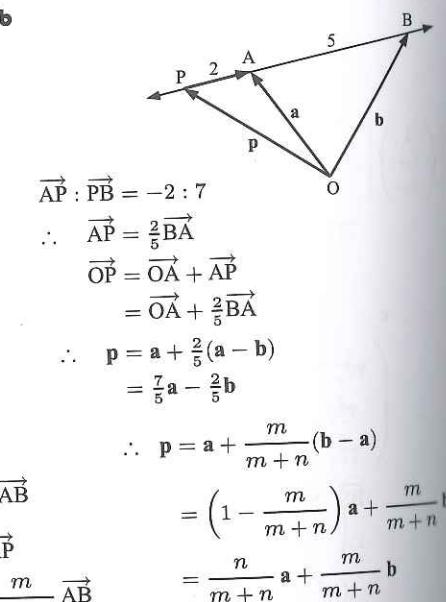
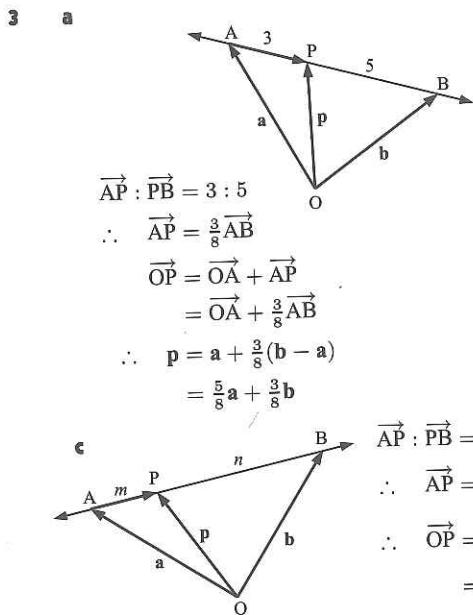
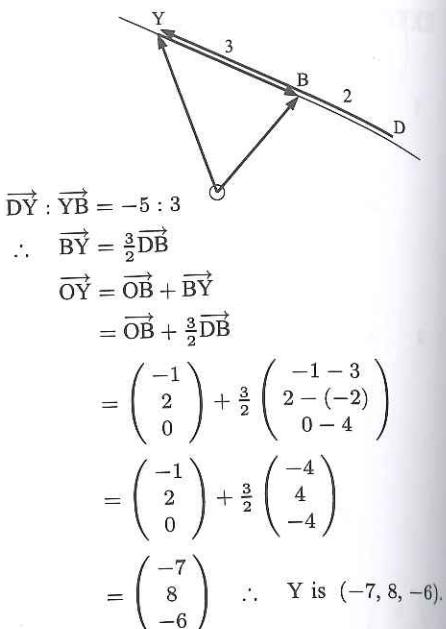
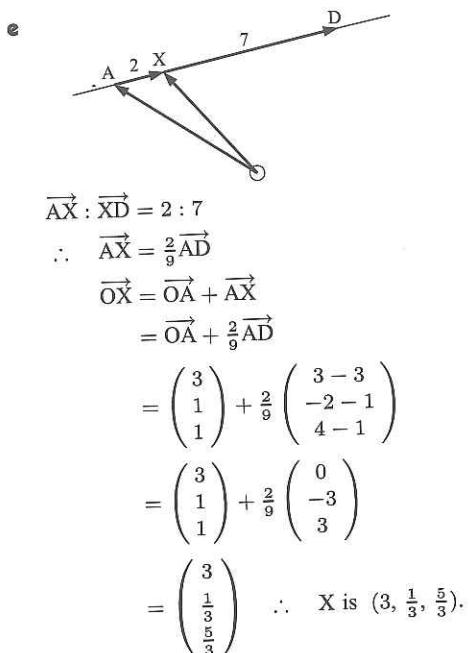
$$\begin{aligned} \overrightarrow{CR} : \overrightarrow{RA} &= -3 : 4 \\ \therefore \overrightarrow{CR} &= 3 \overrightarrow{AC} \\ \overrightarrow{OR} &= \overrightarrow{OC} + \overrightarrow{CR} \\ &= \overrightarrow{OC} + 3 \overrightarrow{AC} \\ &= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} 1 - (-1) \\ -1 - 1 \\ 4 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ -7 \\ 13 \end{pmatrix} \quad \therefore R \text{ is } (-5, -7, 13). \end{aligned}$$



$$\begin{aligned} \overrightarrow{BS} : \overrightarrow{SA} &= 3 : 1 \\ \therefore \overrightarrow{BS} &= \frac{3}{4} \overrightarrow{BA} \\ \overrightarrow{OS} &= \overrightarrow{OB} + \overrightarrow{BS} \\ &= \overrightarrow{OB} + \frac{3}{4} \overrightarrow{BA} \\ &= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 3 - (-1) \\ 1 - 2 \\ 1 - 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{4} \\ \frac{3}{4} \\ \frac{3}{4} \end{pmatrix} \quad \therefore S \text{ is } \left(\frac{5}{4}, \frac{3}{4}, \frac{3}{4}\right). \end{aligned}$$

c $\overrightarrow{CT} : \overrightarrow{TB} = -2 : 5$
 $\therefore \overrightarrow{CT} = \frac{2}{3} \overrightarrow{BC}$

$$\begin{aligned} \overrightarrow{OT} &= \overrightarrow{OC} + \overrightarrow{CT} \\ &= \overrightarrow{OC} + \frac{2}{3} \overrightarrow{BC} \\ &= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1 - (-1) \\ -1 - 1 \\ 4 - 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{7}{3} \\ -3 \\ \frac{20}{3} \end{pmatrix} \quad \therefore T \text{ is } \left(\frac{7}{3}, -3, \frac{20}{3}\right). \end{aligned}$$

**EXERCISE 14G**

1 Since \mathbf{a} and \mathbf{b} are parallel, then $\mathbf{b} = k\mathbf{a}$. $\therefore \begin{pmatrix} -6 \\ r \\ s \end{pmatrix} = k \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2k \\ -k \\ 3k \end{pmatrix}$

$$\therefore 2k = -6, r = -k, s = 3k \quad \therefore k = -3, r = 3, s = -9$$

2 If $\begin{pmatrix} a \\ 2 \\ b \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ are parallel, then $\begin{pmatrix} a \\ 2 \\ b \end{pmatrix} = k \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

$$\therefore 2 = -k, a = 3k, b = 2k \quad \therefore k = -2, a = -6 \text{ and } b = -4$$

3 a Let the vector parallel to \mathbf{a} be $k\mathbf{a}$.

$$\therefore k\mathbf{a} = k \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2k \\ -k \\ -2k \end{pmatrix}$$

Now $k\mathbf{a}$ has length = 1,

$$\text{so } \sqrt{(2k)^2 + (-k)^2 + (-2k)^2} = 1 \\ \therefore 4k^2 + k^2 + 4k^2 = 1$$

$$\therefore 9k^2 = 1$$

$$\therefore k = \pm \frac{1}{3}$$

Choosing $k = \frac{1}{3}$, the vector is $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$.

b Let the vector parallel to \mathbf{b} be $k\mathbf{b}$.

$$\therefore k\mathbf{b} = k \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2k \\ -k \\ 2k \end{pmatrix}$$

Now $k\mathbf{b}$ has length = 2,

$$\text{so } \sqrt{(-2k)^2 + (-k)^2 + (2k)^2} = 2 \\ \therefore 4k^2 + k^2 + 4k^2 = 4$$

$$\therefore 9k^2 = 4$$

$$\therefore k = \pm \frac{2}{3}$$

Choosing $k = \frac{2}{3}$, the vector is $\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$.

4 a $\overrightarrow{AB} = 3\overrightarrow{CD}$ means that \overrightarrow{AB} is parallel to \overrightarrow{CD} and 3 times its length.



$\overrightarrow{AB} = 2\overrightarrow{BC}$ means that A, B and C are collinear and the length of \overrightarrow{AB} is twice the length of \overrightarrow{BC} .

5 $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}, \overrightarrow{OR} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \overrightarrow{OS} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

a $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$

$\overrightarrow{QS} = \overrightarrow{QO} + \overrightarrow{OS} = -\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 2\overrightarrow{PR}$ and so $[QS] \parallel [PR]$.

b Since $\overrightarrow{QS} = 2\overrightarrow{PR}$, $|\overrightarrow{QS}| = 2|\overrightarrow{PR}|$, and so $[QS]$ is twice as long as $[PR]$.

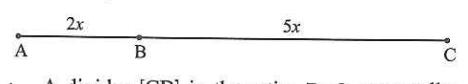
6 a $\overrightarrow{AB} = \begin{pmatrix} 4 - (-2) \\ 3 - 1 \\ 0 - 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 19 - 4 \\ 8 - 3 \\ -10 - 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

b $\overrightarrow{RP} = \begin{pmatrix} 2 - (-1) \\ 1 - 7 \\ 1 - 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

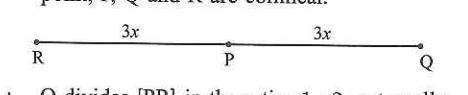
$\overrightarrow{PQ} = \begin{pmatrix} 5 - 2 \\ -5 - 1 \\ -2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

\overrightarrow{AB} is parallel to \overrightarrow{BC} , and since B is a common point, A, B and C are collinear.



A divides $[CB]$ in the ratio 7 : 2 externally.

\overrightarrow{RP} is parallel to \overrightarrow{PQ} , and since P is a common point, P, Q and R are collinear.



Q divides $[PR]$ in the ratio 1 : 2 externally.

7 a

Since A, B and C are collinear, \overrightarrow{CA} is parallel to \overrightarrow{AB} .

$$\therefore \begin{pmatrix} -15 \\ -3-a \\ 4-b \end{pmatrix} = k \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$$

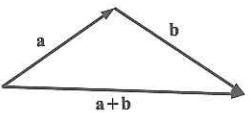
$$\therefore \begin{cases} 15 = 9k \\ -3-a = -6k \\ 4-b = 3k \end{cases}$$

$$\therefore k = \frac{5}{3}$$

$$\therefore a = -3 + 6k = -3 + 10 = 7$$

$$\therefore b = 4 - 3k = 4 - 5 = -1$$

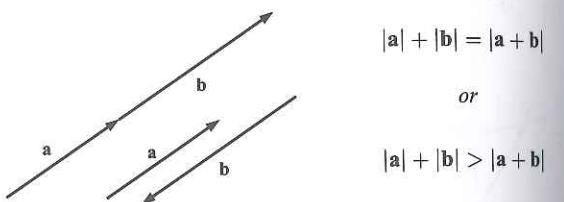
8 • Consider \mathbf{a} not parallel to \mathbf{b} :



Clearly, $|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$

- If $\mathbf{a} = 0$, $\mathbf{b} \neq 0$, then $\mathbf{a} + \mathbf{b} = \mathbf{b}$ $\therefore |\mathbf{a}| + |\mathbf{b}| = 0 + |\mathbf{b}| = |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$
Similarly if $\mathbf{a} \neq 0$, $\mathbf{b} = 0$, then $\mathbf{a} + \mathbf{b} = \mathbf{a}$
 $\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}| + 0 = |\mathbf{a}| = |\mathbf{a} + \mathbf{b}|$
- If $\mathbf{a} = 0$ and $\mathbf{b} = 0$, then $\mathbf{a} + \mathbf{b} = 0$
 $\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

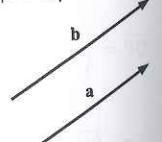
Combining all possibilities, $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$, or $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$



$$|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$$

or

$$|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$$



EXERCISE 14H

1 a $\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
 $\therefore |\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{3}$ units

c $\mathbf{i} - 5\mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$
 $\therefore |\mathbf{i} - 5\mathbf{k}| = \sqrt{1+25} = \sqrt{26}$ units

2 a length = 1
 $\therefore \sqrt{0^2 + k^2} = 1$
 $\therefore k^2 = 1$
 $\therefore k = \pm 1$

b $3\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$
 $\therefore |3\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{9+1+1} = \sqrt{11}$ units

d $\frac{1}{2}(\mathbf{j} + \mathbf{k}) = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$
 $\therefore \left| \frac{1}{2}(\mathbf{j} + \mathbf{k}) \right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ units

b length = 1
 $\therefore \sqrt{k^2 + 0} = 1$
 $\therefore k^2 = 1$
 $\therefore k = \pm 1$

c length = 1
 $\therefore \sqrt{k^2 + 1} = 1$
 $\therefore k^2 + 1 = 1$
 $\therefore k^2 = 0$
 $\therefore k = 0$

d length = 1
 $\therefore \sqrt{\left(\frac{1}{4}\right) + k^2 + \frac{1}{16}} = 1$
 $\therefore \sqrt{k^2 + \frac{5}{16}} = 1$
 $\therefore k^2 = \frac{11}{16} \therefore k = \pm \frac{\sqrt{11}}{4}$

3 a length
 $= \sqrt{3^2 + 4^2}$
 $= \sqrt{9+16}$
 $= \sqrt{25}$
 $= 5$ units

b length
 $= \sqrt{2^2 + (-1)^2 + 1^2}$
 $= \sqrt{4+1+1}$
 $= \sqrt{6}$ units

c length
 $= \sqrt{1^2 + 2^2 + (-2)^2}$
 $= \sqrt{1+4+4}$
 $= \sqrt{9}$
 $= 3$ units

d length = $\sqrt{(-2.36)^2 + (5.65)^2} \approx 6.12$ units

4 a $\mathbf{i} + 2\mathbf{j}$ has length $\sqrt{1^2 + 2^2} = \sqrt{5}$ units \therefore unit vector = $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$

b $2\mathbf{i} - 3\mathbf{k}$ has length $\sqrt{2^2 + 0^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$ units
 \therefore unit vector is $\frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{k})$

c $-2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ has length $\sqrt{(-2)^2 + (-5)^2 + (-2)^2} = \sqrt{4+25+4} = \sqrt{33}$ units
 \therefore unit vector is $\frac{1}{\sqrt{33}}(-2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$

5 a $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ has length $\sqrt{2^2 + (-1)^2} = \sqrt{5}$ units

\therefore the unit vector in the same direction is $\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

\therefore the vector of length 3 units in the same direction is $\frac{3}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$

b $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ has length $\sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$ units

\therefore the unit vector in the opposite direction is $-\frac{1}{\sqrt{17}} \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{\sqrt{17}} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

\therefore the vector of length 2 units in the opposite direction is $-\frac{2}{\sqrt{17}} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{pmatrix}$

c $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ has length $\sqrt{(-1)^2 + 4^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$ units

\therefore the unit vector in the same direction is $\frac{1}{3\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

\therefore the vector of length 6 units in the same direction is $\frac{6}{3\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

d $\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$ has length $\sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$ units

\therefore the unit vector in the opposite direction is $-\frac{1}{3} \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

\therefore the vector of length 5 units in the opposite direction is $\frac{5}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{pmatrix}$

EXERCISE 14I

1 a $\mathbf{q} \bullet \mathbf{p}$
 $= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
 $= -3 + 10$
 $= 7$

b $\mathbf{q} \bullet \mathbf{r}$
 $= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix}$
 $= 2 + 20$
 $= 22$

c $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$
 $= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right]$
 $= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 6 \end{pmatrix}$
 $= -1 + 30 = 29$

d $3\mathbf{r} \bullet \mathbf{q}$
 $= 3 \begin{pmatrix} -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
 $= \begin{pmatrix} -6 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
 $= 6 + 60 = 66$

e $2\mathbf{p} \bullet 2\mathbf{p}$
 $= 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \bullet 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
 $= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 4 \end{pmatrix}$
 $= 36 + 16 = 52$

f $\mathbf{i} \bullet \mathbf{p}$
 $= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
 $= 3 + 0$
 $= 3$

g $\mathbf{q} \bullet \mathbf{j}$
 $= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
 $= 0 + 5$
 $= 5$

h $\mathbf{i} \bullet \mathbf{i}$
 $= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
 $= 1 + 0$
 $= 1$

2 a $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$
 $= 2(-1) + 1(1) + 3(1)$
 $= -2 + 1 + 3$
 $= 2$

c $|\mathbf{a}|^2 = \left(\sqrt{2^2 + 1^2 + 3^2} \right)^2$
 $= 14$

e $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$
 $= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \left[\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right]$
 $= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$
 $= 2(-1) + 1(0) + 3(2) = 4$

3 a $\mathbf{p} \bullet \mathbf{q}$
 $= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$
 $= -6 - 1 + 6$
 $= -1$

b If the angle between \mathbf{p} and \mathbf{q} is θ , then
 $\cos \theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}| |\mathbf{q}|} = \frac{-1}{\sqrt{3^2 + (-1)^2 + 2^2} \sqrt{(-2)^2 + 1^2 + 3^2}}$
 $= \frac{-1}{\sqrt{14} \sqrt{14}}$
 $\therefore \theta = \arccos\left(-\frac{1}{14}\right) \approx 94.1^\circ$

4 a $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k})$
 $= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$
 $= 1(0) + 1(2) - 1(1) = 1$

b $\mathbf{i} \bullet \mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$

c $\mathbf{i} \bullet \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

$$\therefore \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d}$$

If we let $\mathbf{p} = \mathbf{a} + \mathbf{b}$,
then $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d})$
 $= \mathbf{p} \bullet (\mathbf{c} + \mathbf{d})$
 $= \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d}$
 $= (\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} + (\mathbf{a} + \mathbf{b}) \bullet \mathbf{d}$
 $= \mathbf{c} \bullet (\mathbf{a} + \mathbf{b}) + \mathbf{d} \bullet (\mathbf{a} + \mathbf{b})$
 $= \mathbf{c} \bullet \mathbf{a} + \mathbf{c} \bullet \mathbf{b} + \mathbf{d} \bullet \mathbf{a} + \mathbf{d} \bullet \mathbf{b}$
 $= \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$

5 $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$
 $= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right]$
 $= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{pmatrix}$
 $= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$
 $= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$
 $= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$
 $= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

6 a $\begin{pmatrix} 3 \\ t \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0$
 $\therefore -6 + t = 0$
 $\therefore t = 6$

b $\begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 0$
 $\therefore 3t - 4(t+2) = 0$
 $\therefore 3t - 4t - 8 = 0$
 $\therefore -t = 8$
 $\therefore t = -8$

c $\begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 2-t \\ t \end{pmatrix} = 0$
 $\therefore 2t - 3t^2 + t^2 + 2t = 0$
 $\therefore -2t^2 + 4t = 0$
 $\therefore t^2 - 2t = 0$
 $\therefore t(t-2) = 0$
 $\therefore t = 0 \text{ or } 2$

d $\begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix} \bullet \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix} = 0$
 $\therefore 3(2t) + (-1)(-3) + t(-4) = 0$
 $\therefore 6t + 3 - 4t = 0$
 $\therefore 2t + 3 = 0 \quad \therefore t = -\frac{3}{2}$

7 a If $\mathbf{p} \parallel \mathbf{q}$ then $\begin{pmatrix} 3 \\ t \end{pmatrix} = k \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ where $k \neq 0$
 $\therefore 3 = -2k \quad \text{and} \quad t = k$
 $\therefore k = -\frac{3}{2} \quad \text{and} \quad t = -\frac{3}{2}$

b If $\mathbf{r} \parallel \mathbf{s}$ then $\begin{pmatrix} t \\ t+2 \end{pmatrix} = k \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ where $k \neq 0$
 $\therefore t = 3k \quad \text{and} \quad t+2 = -4k$
 $\therefore t+2 = -4\left(\frac{t}{3}\right)$
 $\therefore 3t + 6 = -4t$
 $\therefore 7t = -6$
 $\therefore t = -\frac{6}{7}$

c If $\mathbf{a} \parallel \mathbf{b}$ then $\begin{pmatrix} t \\ t+2 \end{pmatrix} = k \begin{pmatrix} 2-t \\ t \end{pmatrix}$
 $\therefore t = k(2-3t) \quad \text{and} \quad t+2 = kt$
 $\therefore \frac{t}{2-3t} = \frac{t+2}{t} \quad \{ \text{equating } ks \}$
 $\therefore t^2 = (t+2)(2-3t)$
 $\therefore t^2 = 2t - 3t^2 + 4 - 6t$
 $\therefore 4t^2 + 4t - 4 = 0$
 $\therefore t^2 + t - 1 = 0 \quad \text{which has } \Delta = 1^2 - 4(1)(-1) = 5$
 $\therefore t = \frac{-1 \pm \sqrt{5}}{2}$

- d** If $\mathbf{a} \parallel \mathbf{b}$ then $\begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix} = k \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix}$ where $k \neq 0$
- $$\therefore 3 = 2kt, -1 = -3k \text{ and } t = -4k$$
- $$\therefore k = \frac{1}{3} \text{ and } 3 = \frac{2}{3}t, t = -\frac{4}{3}$$
- $$\therefore t = \frac{9}{2} \text{ and } -\frac{4}{3} \text{ simultaneously which is impossible.}$$
- $$\therefore \text{the vectors can never be parallel.}$$

8

a $\mathbf{a} \bullet \mathbf{b}$

$$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$= 3(-1) + 1(1) + 2(1)$$

$$= 0$$

b $\mathbf{b} \bullet \mathbf{c}$

$$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$$

$$= (-1)(1) + 1(5) + 1(-4)$$

$$= 0$$

c $\mathbf{a} \bullet \mathbf{c}$

$$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$$

$$= (3)(1) + 1(5) + 2(-4)$$

$$= 0$$

$\therefore \mathbf{a}, \mathbf{b}$ and \mathbf{c} are mutually perpendicular.

9

a $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 1(2) + 1(3) + 5(-1)$

$$= 0$$

$\therefore \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are perpendicular.

b $\begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix} = 0$

$$\therefore 3(1-t) + t(-3) + (-2)4 = 0$$

$$\therefore 3 - 3t - 3t - 8 = 0$$

$$\therefore -6t = 5$$

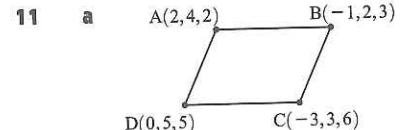
$$\therefore t = -\frac{5}{6}$$

10 We have three points: A(5, 1, 2), B(6, -1, 0), C(3, 2, 0)

Then $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = (-2) + (-2) + 4 = 0$

$\therefore \overrightarrow{AB}$ is perpendicular to \overrightarrow{AC} and so $\triangle ABC$ is right angled at A.

11

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ $\therefore \overrightarrow{AB}$ is parallel to \overrightarrow{DC} and \overrightarrow{BC} is parallel to \overrightarrow{AD} .

$\overrightarrow{DC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ $\therefore ABCD$ is a parallelogram.

b $|\overrightarrow{AB}| = \sqrt{14}$ units and $|\overrightarrow{BC}| = \sqrt{14}$ units $\therefore ABCD$ is a rhombus.

c $\overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (-5) \times 1 + (-1) \times 3 + 4(2) = 0$

$\therefore \overrightarrow{AC}$ is perpendicular to \overrightarrow{BD} which illustrates that the diagonals of a rhombus are perpendicular.

- 12** **a** $x - y = 3$ has gradient $+1$ and so has direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
 $3x + 2y = 11$ has gradient $-\frac{3}{2}$ and so has direction vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.
- $$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \sqrt{1+1}\sqrt{4+9}\cos\theta$$
- $$\therefore 2 - 3 = \sqrt{2}\sqrt{13}\cos\theta$$
- $$\therefore \frac{-1}{\sqrt{26}} = \cos\theta$$
- $$\therefore \theta \approx 101.3^\circ \therefore \text{the angle is } 101.3^\circ \text{ or } 78.7^\circ$$

- b** $y = x + 2$ has slope $1 = \frac{1}{1}$ \therefore direction vector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
 $y = 1 - 3x$ has slope $-3 = \frac{-3}{1}$ \therefore direction vector is $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \sqrt{1+1}\sqrt{1+9}\cos\theta$$

$$\therefore 1 - 3 = \sqrt{2}\sqrt{10}\cos\theta$$

$$\therefore \frac{-2}{\sqrt{20}} = \cos\theta$$

$$\therefore \theta \approx 116.6^\circ \therefore \text{the angle is } 116.6^\circ \text{ or } 63.4^\circ$$

- c** $y + x = 7$ has slope $-1 = \frac{-1}{1}$ \therefore direction vector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
 $x - 3y + 2 = 0$ has slope $\frac{1}{3}$ \therefore direction vector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \sqrt{1+1}\sqrt{9+1}\cos\theta$$

$$\therefore 3 - 1 = \sqrt{2}\sqrt{10}\cos\theta$$

$$\therefore \frac{2}{\sqrt{20}} = \cos\theta$$

$$\therefore \theta \approx 63.4^\circ \therefore \text{the angle is } 63.4^\circ \text{ or } 116.6^\circ$$

- d** $y = 2 - x$ has slope $-1 = \frac{-1}{1}$ \therefore has direction vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
 $x - 2y = 7$ has slope $\frac{1}{2}$ \therefore has direction vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \sqrt{1+1}\sqrt{4+1}\cos\theta$$

$$\therefore 2 - 1 = \sqrt{2}\sqrt{5}\cos\theta$$

$$\therefore \cos\theta = \frac{1}{\sqrt{10}}$$

$$\therefore \theta \approx 71.6^\circ \therefore \text{the angle is } 71.6^\circ \text{ or } 108.4^\circ$$

13 **a** $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos\theta$

$$= 2 \times 5 \times \cos 60^\circ$$

$$= 5$$

b $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos\theta$

$$= 6 \times 3 \times \cos 120^\circ$$

$$= -9$$

- 14** **a** $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} = -10 + 10 = 0$, so $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is one such vector.

\therefore required vectors have form $k\begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $k \neq 0$. Note: $k\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $k \neq 0$ is also acceptable.

- b** $\begin{pmatrix} -1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 2 - 2 = 0$, so $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is one such vector.

\therefore required vectors have form $k\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $k \neq 0$.

- c** $\begin{pmatrix} 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 3 - 3 = 0$, so $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is one such vector.

\therefore required vectors have form $k\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $k \neq 0$.

- d** $\begin{pmatrix} -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \end{pmatrix} = -12 + 12 = 0$, so $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is one such vector.

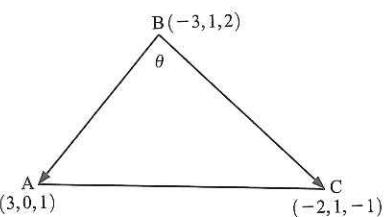
\therefore required vectors have form $k\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $k \neq 0$.

- e** $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 + 0 = 0$, so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is one such vector.

\therefore required vectors have form $k\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $k \neq 0$.

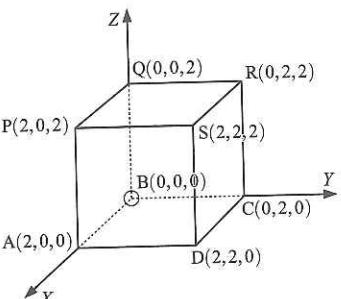
- 15 Given $A(3, 0, 1)$, $B(-3, 1, 2)$ and $C(-2, 1, -1)$,

$$\vec{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \text{ and } \vec{BA} = \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}$$



If \vec{BA} and \vec{CB} are used we would find the exterior angle of the triangle at B, which is 117.5° .

16



- a Suppose the origin is at B.

$$\text{Now } \vec{BA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \text{ and } \vec{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\therefore \vec{BA} \cdot \vec{BS} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 4 + 0 + 0 = 4$$

$$\therefore \cos \hat{ABS} = \frac{4}{\sqrt{4+0+0}\sqrt{4+4+4}} = \frac{4}{2 \times 2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore \hat{ABS} \approx 54.7^\circ$$

- b Consider vectors away from B.

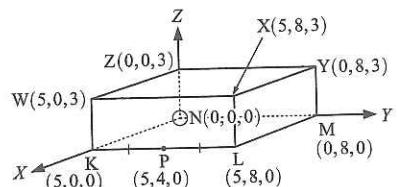
$$\vec{BR} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \text{ and } \vec{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$\vec{BR} \cdot \vec{BP} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 0 + 0 + 4 = 4$$

$$\therefore \cos \hat{RBP} = \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}} = \frac{4}{\sqrt{8} \times \sqrt{8}} = \frac{1}{2} \text{ and so } \hat{RBP} = 60^\circ$$

- 17 Suppose the origin is at N.

a



$$\vec{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \text{ and } \vec{NX} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}$$

$$\vec{NY} \cdot \vec{NX} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} = 0 + 64 + 9 = 73$$

$$\therefore \cos \hat{YNX} = \frac{73}{\sqrt{64+9}\sqrt{25+64+9}} = \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\frac{73}{98}}$$

$$\therefore \hat{YNX} \approx 30.3^\circ$$

b $\vec{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix}$ and $\vec{NP} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

$$\begin{aligned} \vec{NY} \cdot \vec{NP} &= \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \\ &= 0 + 32 + 0 \\ &= 32 \end{aligned}$$

$$\therefore \cos \hat{YNP} = \frac{32}{\sqrt{64+9}\sqrt{25+16}} = \frac{32}{\sqrt{73}\sqrt{41}} \approx 54.2^\circ$$

- 18 a M is the midpoint of [BC]. \therefore M is at $\left(\frac{2+1}{2}, \frac{2+3}{2}, \frac{2+1}{2}\right)$, which is $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$.

b Now $\vec{MD} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$ and $\vec{MA} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$

$$\therefore \cos \theta = \frac{\vec{MD} \cdot \vec{MA}}{|\vec{MD}| |\vec{MA}|} = \frac{\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4} + \frac{9}{4}} \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}}$$

$$\therefore \cos \theta = \frac{\frac{3}{4} + \frac{3}{4} + \frac{3}{4}}{\sqrt{\frac{19}{4}} \sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\sqrt{209}} = \frac{9}{\sqrt{209}} \text{ and so } \theta \approx 51.5^\circ$$

19 a $\begin{pmatrix} 2 \\ t \\ t-2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ t \\ t \end{pmatrix} = 0 \quad \therefore 2t + 3t + t(t-2) = 0$
 $\therefore 5t + t^2 - 2t = 0$
 $\therefore t^2 + 3t = 0$
 $\therefore t(t+3) = 0 \text{ and so } t = 0 \text{ or } t = -3$

- b Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ are mutually perpendicular,
 $\mathbf{a} \cdot \mathbf{b} = 0$, $\mathbf{b} \cdot \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{c} = 0$

$$\therefore \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix} = 0 \quad \therefore 2 + 4 + 3r = 0$$

$$\therefore 3r = -6 \quad \therefore r = -2$$

$$\text{and } \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \quad \therefore 2s + 2t - 2 = 0$$

$$\therefore s + t = 1 \quad \dots\dots (1)$$

$$\text{and } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \quad \therefore s + 2t + 3 = 0$$

$$\therefore s + 2t = -3 \quad \dots\dots (2)$$

$$(2) - (1) \text{ gives } t = -4 \text{ and so } s = 5 \quad \therefore r = -2, s = 5 \text{ and } t = -4$$

- 10 a Choose any vector in the direction of the X-axis, such as $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\text{Then } \cos \theta = \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|} = \frac{1}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}} \text{ and so } \theta \approx 74.5^\circ$$

b A line parallel to the Y -axis has direction vector $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

$$\text{Then } \cos \theta = \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}}{\sqrt{1+1+9}} = \frac{1}{\sqrt{11}} \text{ and so } \theta \approx 72.5^\circ.$$

21 We want vectors \mathbf{a} , \mathbf{b} and \mathbf{c} such that $\mathbf{a} \neq \mathbf{0}$, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \neq \mathbf{c}$

$$\text{For example, } \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

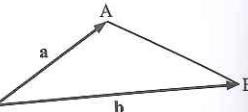
In this case, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0$

$$\begin{aligned} 22 \quad \mathbf{a} \cdot |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= 2\mathbf{a} \cdot \mathbf{a} + 2\mathbf{b} \cdot \mathbf{b} \\ &= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 \quad \text{as required} \end{aligned}$$

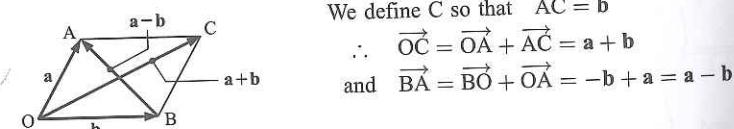
$$\begin{aligned} \mathbf{b} \quad |\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) - (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\ &= 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{b} \cdot \mathbf{a} \\ &= 4\mathbf{a} \cdot \mathbf{b} \quad \text{as required } \{ \text{since } \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \} \end{aligned}$$

23 We are given that $\mathbf{a} \neq \mathbf{b}$, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$

$$\begin{aligned} \mathbf{a} \quad &\text{Now if } |\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}| \\ &\text{then } |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2 \\ &\therefore (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &\therefore \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &\therefore 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{b} \cdot \mathbf{a} = 0 \\ &\therefore 4\mathbf{a} \cdot \mathbf{b} = 0 \quad \{ \text{as } \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \} \\ &\therefore \mathbf{a} \cdot \mathbf{b} = 0, \text{ and since neither } \mathbf{a} \text{ nor } \mathbf{b} = \mathbf{0} \\ &\quad \mathbf{a} \text{ is perpendicular to } \mathbf{b}. \end{aligned}$$



b Consider the following diagram representing \mathbf{a} , \mathbf{b} , $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$:



$$\begin{aligned} \text{We define } \mathbf{C} \text{ so that } \overrightarrow{AC} = \mathbf{b} \\ \therefore \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \mathbf{b} \\ \text{and } \overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b} \end{aligned}$$

$\therefore \mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ represent the diagonals of the parallelogram OACB

But if $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then the diagonals must be equal in length.

This is only possible if OACB is a square or rectangle, which means that \mathbf{a} is perpendicular to \mathbf{b} .

$$\begin{aligned} 24 \quad (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \quad \{ \text{since } \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \} \\ &= |\mathbf{a}|^2 - |\mathbf{b}|^2 \\ &= 9 - 16 \\ &= -7 \end{aligned}$$

25 The scalar product is only defined between two vectors.

Hence $\underbrace{(\mathbf{a} \cdot \mathbf{b})}_{\substack{\uparrow \\ \text{scalar}}} \cdot \mathbf{c}$ or $\mathbf{a} \cdot \underbrace{(\mathbf{b} \cdot \mathbf{c})}_{\substack{\uparrow \\ \text{vector}}} \text{ is meaningless.}$

EXERCISE 14J.1

$$\begin{aligned} 1 \quad \mathbf{a} \quad &\left(\begin{array}{c} 2 \\ -3 \\ 1 \end{array} \right) \times \left(\begin{array}{c} 1 \\ 4 \\ -2 \end{array} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 4 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} \mathbf{k} \\ &= (6 - 4)\mathbf{i} - (-4 - 1)\mathbf{j} + (8 - (-3))\mathbf{k} \\ &= 2\mathbf{i} + 5\mathbf{j} + 11\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\left(\begin{array}{c} -1 \\ 0 \\ 2 \end{array} \right) \times \left(\begin{array}{c} 3 \\ -1 \\ -2 \end{array} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 3 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 0 \\ 3 & -1 \end{vmatrix} \mathbf{k} \\ &= (0 - (-2))\mathbf{i} - (2 - 6)\mathbf{j} + (1 - 0)\mathbf{k} \\ &= 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad &(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\ &= (-1 + 0)\mathbf{i} - (-1 - (-2))\mathbf{j} + (0 - 1)\mathbf{k} \\ &= -\mathbf{i} - \mathbf{j} - \mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad &(2\mathbf{i} - \mathbf{k}) \times (\mathbf{j} + 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= (0 - (-1))\mathbf{i} - (6 - 0)\mathbf{j} + (2 - 0)\mathbf{k} \\ &= \mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \end{aligned}$$

$$2 \quad \text{If } \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

$$\text{then } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \mathbf{k} \\ = -11\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

$$\therefore \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -11 \\ -2 \\ 5 \end{pmatrix} = -11 - 4 + 15 = 0$$

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -11 \\ -2 \\ 5 \end{pmatrix} = 11 - 6 - 5 = 0$$

$\therefore \mathbf{a} \times \mathbf{b}$ is perpendicular to both \mathbf{a} and \mathbf{b} .

$$3 \quad \mathbf{a} \quad \mathbf{i} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0, \quad \mathbf{j} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0, \quad \mathbf{k} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\mathbf{b} \quad \mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$$

$$\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}, \quad \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$$

If \mathbf{a} and \mathbf{b} are vectors then $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

4 $\mathbf{a} \cdot \mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} \mathbf{k}$
 $= 0 \times \mathbf{i} + 0 \times \mathbf{j} + 0 \times \mathbf{k}$
 $= \mathbf{0}$

b $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$
 $= (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}$
 $= -[(a_3 b_2 - a_2 b_3) \mathbf{i} - (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_2 b_1 - a_1 b_2) \mathbf{k}]$
 $= -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$
 $= -\mathbf{b} \times \mathbf{a}$

5 $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \mathbf{k}$
 $= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$
 $= \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

b $\mathbf{a} \circ (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 1 + 12 + 4 = 17$

c $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$
 $= 1(1) - 3(-4) + 2(2)$
 $= 1 + 12 + 4$
 $= 17$

7 $\mathbf{a} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \mathbf{k}$
 $= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

b $\mathbf{a} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{k}$
 $= 0\mathbf{i} + 5\mathbf{j} + 0\mathbf{k}$
 $= 5\mathbf{j}$

c $(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + 5\mathbf{j}$ {using **a** and **b**}
 $= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

d $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times (2\mathbf{i} - \mathbf{j})$
 $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \mathbf{k}$
 $= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

8 We suspect that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

9 $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$
 $= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$
 $= \begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 + c_1 & b_3 + c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 + c_1 & b_2 + c_2 \end{vmatrix} \mathbf{k}$
 $= (a_2(b_3 + c_3) - a_3(b_2 + c_2))\mathbf{i} - (a_1(b_3 + c_3) - a_3(b_1 + c_1))\mathbf{j} + (a_1(b_2 + c_2) - a_2(b_1 + c_1))\mathbf{k}$
 $= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
 $+ (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}$
 $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
 $= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ as required

10 Now $\mathbf{p} \times (\mathbf{c} + \mathbf{d}) = \mathbf{p} \times \mathbf{c} + \mathbf{p} \times \mathbf{d}$
 \therefore if we let $\mathbf{p} = (\mathbf{a} + \mathbf{b})$,
then $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{p} \times (\mathbf{c} + \mathbf{d})$
 $= \mathbf{p} \times \mathbf{c} + \mathbf{p} \times \mathbf{d}$
 $= (\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b}) \times \mathbf{d}$
 $= -\mathbf{c} \times (\mathbf{a} + \mathbf{b}) - \mathbf{d} \times (\mathbf{a} + \mathbf{b})$ {since $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$ }

11 $\mathbf{a} \cdot \mathbf{a} \times (\mathbf{a} + \mathbf{b})$
 $= \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}$
 $= \mathbf{0} + \mathbf{a} \times \mathbf{b}$
 $= \mathbf{a} \times \mathbf{b}$

b $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b})$
 $= \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b}$
 $= \mathbf{0} + \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{0}$
 $= \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a}$
 $= \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b}$
 $= \mathbf{0}$

c $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$
 $= \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b}$
 $= \mathbf{0} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{a} - \mathbf{0}$
 $= 2(\mathbf{b} \times \mathbf{a})$

d \mathbf{a} is perpendicular to $(\mathbf{a} \times \mathbf{b})$,
 $2\mathbf{a}$ is perpendicular to $(\mathbf{a} \times \mathbf{b})$.
 $\therefore 2\mathbf{a} \circ (\mathbf{a} \times \mathbf{b}) = 0$

12 $\mathbf{a} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} = -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

\therefore the vectors are $k(-4\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, $k \neq 0$, $k \in \mathbb{R}$.

b $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 4 \\ 5 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix} \mathbf{k} = 6\mathbf{i} + 22\mathbf{j} - 15\mathbf{k}$

\therefore the vectors are $k(6\mathbf{i} + 22\mathbf{j} - 15\mathbf{k})$, $k \neq 0$, $k \in \mathbb{R}$.

c $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{k} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

\therefore the vectors are $n(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})$, $n \neq 0$, $n \in \mathbb{R}$.

d $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{k} = 5\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

\therefore the vectors are $n(5\mathbf{i} + \mathbf{j} + 4\mathbf{k})$, $n \neq 0$, $n \in \mathbb{R}$.

13 $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \mathbf{k} = 4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$

\therefore the vectors $k(4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k})$, $k \neq 0$ are all perpendicular to both \mathbf{a} and \mathbf{b} .

However, we require the vector to have length 5.

$$\therefore \sqrt{(4k)^2 + (-5k)^2 + (-7k)^2} = 5$$

$$\therefore 16k^2 + 25k^2 + 49k^2 = 25$$

$$\therefore 90k^2 = 25$$

$$\therefore k^2 = \frac{25}{90} = \frac{5}{18}$$

$$\therefore k = \pm \frac{\sqrt{5}}{3\sqrt{2}} = \pm \frac{\sqrt{10}}{6}$$

$$\therefore \text{the possible vectors are } \pm \frac{\sqrt{10}}{6} \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}.$$

14 **a** Given A(1, 3, 2), B(0, 2, -5) and C(3, 1, -4), $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -1 \\ -7 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}$.

$$\therefore \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -7 \\ 2 & -2 & -6 \end{vmatrix} = \begin{vmatrix} -1 & -7 \\ -2 & -6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -7 \\ 2 & -6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} \mathbf{k}$$

$$= (6 - 14)\mathbf{i} - (6 - -14)\mathbf{j} + (2 - -2)\mathbf{k}$$

$$= -8\mathbf{i} - 20\mathbf{j} + 4\mathbf{k}$$

$$= -4(2\mathbf{i} + 5\mathbf{j} - \mathbf{k})$$

$\therefore 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ is one vector perpendicular to the plane.

b Given P(2, 0, -1), Q(0, 1, 3) and R(1, -1, 1), $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$$\therefore \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix} \mathbf{k}$$

$$= 6\mathbf{i} + 3\mathbf{k}$$

$= 3(2\mathbf{i} + \mathbf{k})$ and so $2\mathbf{i} + \mathbf{k}$ is one vector perpendicular to the plane.

EXERCISE 14J.2

1 **a** $\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{k} = 0\mathbf{i} - \mathbf{j} + 0\mathbf{k} = -\mathbf{j}$

$$\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k} = 0\mathbf{i} + \mathbf{j} + 0\mathbf{k} = \mathbf{j}$$

b Using the RH rule these two results check.

c $\mathbf{i} \times \mathbf{k} = |\mathbf{i}| |\mathbf{k}| \sin 90^\circ \times (-\mathbf{j})$ {RH rule}
 $= 1 \times 1 \times 1 \times (-\mathbf{j})$
 $= -\mathbf{j}$

2 **a** $\mathbf{a} \cdot \mathbf{b} = 2 \times 1 + (-1) \times 0 + 3 \times (-1) = -1$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\ &= \mathbf{i} + 5\mathbf{j} + \mathbf{k} \end{aligned}$$

b $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$

$$= \frac{-1}{\sqrt{4+1+9}\sqrt{1+1}}$$

$$= \frac{-1}{\sqrt{14}\sqrt{2}}$$

$$= \frac{-1}{\sqrt{28}}$$

c $\sin^2 \theta + \cos^2 \theta = 1$

$$\therefore \sin^2 \theta + \frac{1}{28} = 1$$

$$\therefore \sin^2 \theta = \frac{27}{28}$$

$$\therefore \sin \theta = \pm \sqrt{\frac{27}{28}}$$

But $0 \leq \theta \leq \pi$, so $\sin \theta = \sqrt{\frac{27}{28}}$

d $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$

$$\therefore \sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}$$

$$= \frac{\sqrt{1^2 + 5^2 + 1^2}}{\sqrt{14}\sqrt{2}}$$

$$= \sqrt{\frac{27}{28}}$$

3 $\mathbf{a} \times \mathbf{b} = \mathbf{0} \quad \therefore |\mathbf{a}| |\mathbf{b}| \sin \theta \times \mathbf{u} = \mathbf{0}$

$$\therefore |\mathbf{a}| |\mathbf{b}| \sin \theta = 0 \quad \{ \text{since } |\mathbf{a}| \neq 0 \text{ and } |\mathbf{b}| \neq 0, \mathbf{u} \text{ exists and } \neq \mathbf{0} \}$$

$$\therefore \sin \theta = 0 \quad \{ \text{since } |\mathbf{a}| \neq 0 \text{ and } |\mathbf{b}| \neq 0 \}$$

$$\therefore \theta = 0 \text{ or } \pi$$

$\therefore \mathbf{a}$ is parallel to \mathbf{b}

4 **a** $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

b $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{k}$

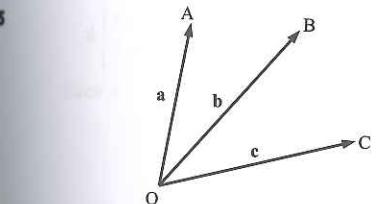
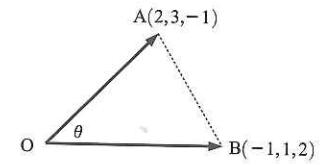
$$= 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$\therefore |\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{7^2 + (-3)^2 + 5^2} = \sqrt{83} \text{ units}$$

c Area $\Delta AOB = \frac{1}{2} |\overrightarrow{OA}| |\overrightarrow{OB}| \sin \theta$

$$= \frac{1}{2} |\overrightarrow{OA} \times \overrightarrow{OB}|$$

$$= \frac{1}{2} \sqrt{83} \text{ units}^2$$



a $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$

$$\therefore \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{c} = \mathbf{0}$$

$$\therefore (\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{0}$$

$$\therefore \overrightarrow{OC} \text{ must be parallel to } \overrightarrow{AB}.$$

c $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$ where $\mathbf{c} \neq \mathbf{0}$

$$\therefore \mathbf{b} \times \mathbf{c} = -\mathbf{a} \times \mathbf{c}$$

$$\therefore \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$$

$$\therefore (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{0}$$

$$\therefore \mathbf{a} + \mathbf{b} \text{ is parallel to } \mathbf{c}$$

$$\therefore \mathbf{a} + \mathbf{b} = k\mathbf{c} \text{ for some scalar } k.$$

EXERCISE 14J.3

1 **a** Given A(2, 1, 1), B(4, 3, 0) and C(1, 3, -2), $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$.

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k}$$

$$= -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

$$\begin{aligned}\therefore \text{area} &= \frac{1}{2} |-4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}| \\ &= \frac{1}{2} \sqrt{(-4)^2 + 7^2 + 6^2} \\ &= \frac{1}{2} \sqrt{101} \text{ units}^2\end{aligned}$$

b Given A(0, 0, 0), B(-1, 2, 3) and C(1, 2, 6), $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$.

$$\begin{aligned}\therefore \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}\end{aligned}$$

$$\begin{aligned}\therefore \text{area} &= \frac{1}{2} |6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}| \quad \{\text{area} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|\} \\ &= \frac{1}{2} \sqrt{6^2 + 9^2 + (-4)^2} \\ &= \frac{1}{2} \sqrt{133} \text{ units}^2\end{aligned}$$

c Given A(1, 3, 2), B(2, -1, 0) and C(1, 10, 6), $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 7 \\ 4 \end{pmatrix}$.

$$\begin{aligned}\therefore \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -2 \\ 0 & 7 & 4 \end{vmatrix} = \begin{vmatrix} -4 & -2 \\ 0 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -4 \\ 0 & 7 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}\end{aligned}$$

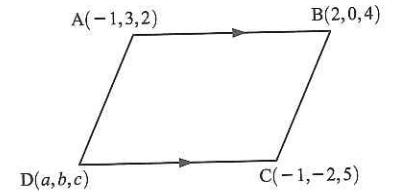
$$\therefore \text{area} = \frac{1}{2} |-2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}| = \frac{1}{2} \sqrt{(-2)^2 + (-4)^2 + 7^2} = \frac{1}{2} \sqrt{69} \text{ units}^2$$

2 Given A(-1, 2, 2), B(2, -1, 4) and C(0, 1, 0), $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

$$\begin{aligned}\therefore \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 2 \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -3 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\ &= 8\mathbf{i} + 8\mathbf{j}\end{aligned}$$

$$\therefore \text{area of parallelogram} = |8\mathbf{i} + 8\mathbf{j}| = \sqrt{8^2 + 8^2} = 8\sqrt{2} \text{ units}^2$$

3



a Suppose D is at (a, b, c) .

Since $\overrightarrow{AB} = \overrightarrow{DC}$,

$$\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1-a \\ -2-b \\ 5-c \end{pmatrix}$$

$$\therefore -1-a=3, -2-b=-3 \text{ and } 5-c=2$$

$$\therefore a=-4, b=1 \text{ and } c=3$$

$$\therefore D \text{ is at } (-4, 1, 3).$$

b $\overrightarrow{BC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$

$$\begin{aligned}\therefore \overrightarrow{BC} \times \overrightarrow{BA} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ -3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ -3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 1 \\ -3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & -2 \\ -3 & 3 \end{vmatrix} \mathbf{k} \\ &= \mathbf{i} - 9\mathbf{j} - 15\mathbf{k}\end{aligned}$$

$$\therefore \text{area} = |\mathbf{i} - 9\mathbf{j} - 15\mathbf{k}| = \sqrt{1^2 + (-9)^2 + (-15)^2} = \sqrt{307} \text{ units}^2$$

4 a Now $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$

\therefore the volume of the tetrahedron

$$\begin{aligned}&= \frac{1}{6} \left| \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) \right| \\ &= \frac{1}{6} \left| \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & -3 \\ -2 & 2 & 2 \end{vmatrix} \right| = \frac{1}{6} \left| \begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} -1 & -3 \\ -2 & 2 \end{vmatrix} \begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix} \right| \\ &= \frac{1}{6} |10 - 2(-8) - 1(2)| \\ &= \frac{1}{6} |24| = 4 \text{ units}^3\end{aligned}$$

b The total surface area of the tetrahedron is the sum of the four triangular faces forming it.

$$\begin{aligned}\text{Face 1 } \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k} \\ &= -4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}\end{aligned}$$

$$\therefore A_1 = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{4^2 + 4^2 + 4^2} = \frac{1}{2} (4\sqrt{3}) = 2\sqrt{3} \text{ units}^2$$

$$\begin{aligned}\text{Face 2 } \overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 6\mathbf{k}\end{aligned}$$

$$\therefore A_2 = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AD}| = \frac{1}{2} \sqrt{6^2 + 6^2} = \frac{1}{2} 6\sqrt{2} = 3\sqrt{2} \text{ units}^2$$

$$\begin{aligned}\text{Face 3 } \overrightarrow{AC} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -3 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -3 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 10\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\therefore A_3 = \frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{AD}| = \frac{1}{2} \sqrt{10^2 + 8^2 + 2^2} = \frac{1}{2} \sqrt{168} = \sqrt{42} \text{ units}^2$$

$$\begin{aligned}\text{Face 4 } \overrightarrow{BC} \times \overrightarrow{BD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -2 \\ -3 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -2 \\ -3 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 0 \\ -3 & 0 \end{vmatrix} \mathbf{k} \\ &= 12\mathbf{j}\end{aligned}$$

$$\therefore A_4 = \frac{1}{2} |\overrightarrow{BC} \times \overrightarrow{BD}| = \frac{1}{2} \times 12 = 6 \text{ units}^2$$

$$\therefore \text{total surface area} = (\sqrt{42} + 2\sqrt{3} + 3\sqrt{2} + 6) \text{ units}^2$$

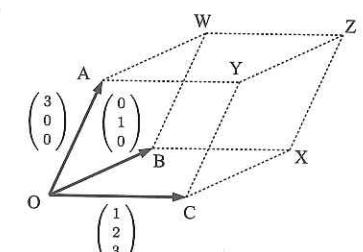
5 a Given A(3, 0, 0), B(0, 1, 0), C(1, 2, 3), O(0, 0, 0), we label the other vertices as shown.

$$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{OC} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

$$\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{OC} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{OW} = \overrightarrow{OA} + \overrightarrow{OB} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OZ} = \overrightarrow{OW} + \overrightarrow{OC} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$$



b) $\overrightarrow{BA} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

$$\therefore \cos A\hat{B}C = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} = \frac{\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}{\sqrt{3^2 + (-1)^2 + 0^2} \sqrt{1^2 + 1^2 + 3^2}} = \frac{2}{\sqrt{110}}$$

$$\therefore \theta \approx 79.0^\circ$$

c) Volume = $\left| \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} \right| = \left| 3 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \right| = |3 \times 3| = 9 \text{ units}^3$

6 Now $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} k+1 \\ 1 \\ -3 \end{pmatrix}$

$$\text{Area of } \triangle ABC = \frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{AB}|$$

$$\therefore \sqrt{88} = \frac{1}{2} \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ k+1 & 1 & -3 \\ 3 & -1 & -1 \end{vmatrix} \right| = \frac{1}{2} \left| \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} k+1 & -3 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} k+1 & 1 \\ 3 & -1 \end{vmatrix} \mathbf{k} \right|$$

$$\therefore \sqrt{352} = |(-1-3)\mathbf{i} - ((k+1)-9)\mathbf{j} + ((-k+1)-3)\mathbf{k}|$$

$$\therefore \sqrt{352} = |-4\mathbf{i} + (k-8)\mathbf{j} + (-k-4)\mathbf{k}|$$

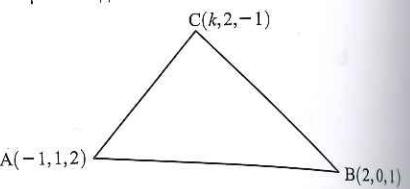
$$\therefore \sqrt{352} = \sqrt{16 + (k-8)^2 + (-k-4)^2}$$

$$\therefore 352 = 16 + k^2 - 16k + 64 + k^2 + 8k + 16$$

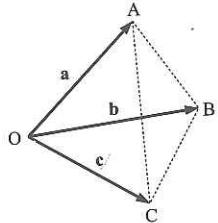
$$\therefore 2k^2 - 8k - 256 = 0$$

$$\therefore k^2 - 4k - 128 = 0$$

$$\therefore k = \frac{4 \pm \sqrt{16 + 4(1)(128)}}{2} = 2 \pm \sqrt{132} = 2 \pm 2\sqrt{33}$$



7

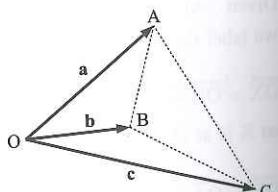


Total surface area S of the tetrahedron is the sum of the areas of the 4 triangular faces.

$$\text{Now } \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

$$\text{and } \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$$

$$\therefore S = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| + \frac{1}{2} |\mathbf{a} \times \mathbf{c}| + \frac{1}{2} |\mathbf{b} \times \mathbf{c}| + \frac{1}{2} |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$$



8 Now $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b}$

and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\mathbf{b} + \mathbf{c} = \mathbf{c} - \mathbf{b}$

$$\therefore \text{area } \triangle ABC = \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}|$$

$$= \frac{1}{2} |(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b})|$$

$$= \frac{1}{2} |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b})|$$

Now if A, B and C are collinear, then $\text{area } \triangle ABC = 0$

$$\therefore (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$$

or $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \mathbf{b} - \mathbf{a}$

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \mathbf{c} - \mathbf{b}$$

If A, B and C are collinear, then \overrightarrow{AB} is parallel to \overrightarrow{BC}

$$\therefore \overrightarrow{AB} \times \overrightarrow{BC} = \mathbf{0} \quad \therefore (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$$

9 $\mathbf{a} \cdot \mathbf{b} - \mathbf{a} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ $\mathbf{c} - \mathbf{a} = \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ $\mathbf{d} - \mathbf{a} = \overrightarrow{AD} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

$$\therefore (\mathbf{b} - \mathbf{a}) \bullet (\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})$$

$$= \begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & -1 \\ 3 & -1 & -1 \end{vmatrix} = 1 \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} - 3 \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$

$$= (-1) - 3(1) - 2(-2)$$

= 0 and so A, B, C and D are coplanar.

10 $\mathbf{q} - \mathbf{p} = \overrightarrow{PQ} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ $\mathbf{r} - \mathbf{p} = \overrightarrow{PR} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$ $\mathbf{s} - \mathbf{p} = \overrightarrow{PS} = \begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix}$

$$\therefore (\mathbf{q} - \mathbf{p}) \bullet (\mathbf{r} - \mathbf{p}) \times (\mathbf{s} - \mathbf{p})$$

$$= \begin{vmatrix} -2 & -1 & -1 \\ 0 & 1 & -5 \\ -1 & 1 & -4 \end{vmatrix} = -2 \begin{vmatrix} 1 & -5 \\ 1 & -4 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -5 \\ -1 & -4 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= -2(1) + 1(-5) - 1(1)$$

$$= -8$$

$\neq 0$ and so P, Q, R and S are not coplanar.

11 $\mathbf{b} - \mathbf{a} = \overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ $\mathbf{c} - \mathbf{a} = \overrightarrow{AC} = \begin{pmatrix} -2 \\ k-1 \\ -1 \end{pmatrix}$ $\mathbf{d} - \mathbf{a} = \overrightarrow{AD} = \begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix}$

$$\therefore (\mathbf{b} - \mathbf{a}) \bullet (\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})$$

$$= \begin{vmatrix} 2 & -1 & -2 \\ -2 & k-1 & -1 \\ -1 & 1 & -4 \end{vmatrix} = 2 \begin{vmatrix} k-1 & -1 \\ 1 & -4 \end{vmatrix} - (-1) \begin{vmatrix} -2 & -1 \\ -1 & -4 \end{vmatrix} - 2 \begin{vmatrix} -2 & k-1 \\ -1 & 1 \end{vmatrix}$$

$$= 2(-4k + 4 + 1) + 1(8 - 1) - 2(-2 + k + 1)$$

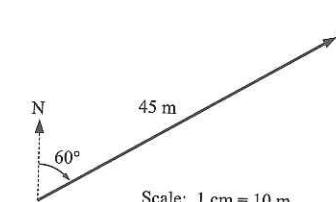
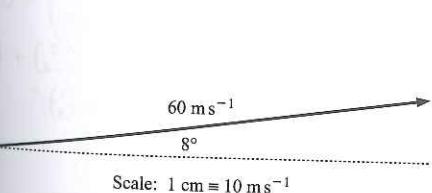
$$= -8k + 10 + 7 - 2k + 6$$

$$= -10k + 23$$

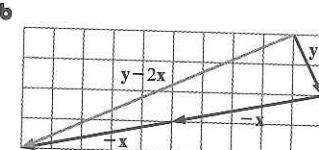
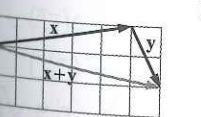
\therefore A, B, C and D are coplanar when $k = \frac{23}{10}$.

REVIEW SET 14A

1 a



1 a



3 a $\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$
b $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

4 Dino's first displacement vector is $9 \begin{pmatrix} \cos 246^\circ \\ \sin 246^\circ \end{pmatrix}$. His second displacement vector is $6 \begin{pmatrix} \cos 96^\circ \\ \sin 96^\circ \end{pmatrix}$.

$$\therefore \text{Dino's resultant displacement vector is } \begin{pmatrix} 9 \cos 246^\circ \\ 9 \sin 246^\circ \end{pmatrix} + \begin{pmatrix} 6 \cos 96^\circ \\ 6 \sin 96^\circ \end{pmatrix} \approx \begin{pmatrix} -4.288 \\ -2.255 \end{pmatrix}$$

which has length $\sqrt{(-4.288)^2 + (-2.255)^2} \approx 4.845$.

$$\therefore \text{the resultant displacement vector is } 4.845 \begin{pmatrix} -0.8851 \\ -0.4654 \end{pmatrix} \begin{matrix} \leftarrow \cos \theta \\ \leftarrow \sin \theta \end{matrix}$$

If $\cos \theta = -0.8850$ and $\sin \theta = -0.4654$, θ is in Quadrant 3

$$\therefore \theta = 180^\circ + \cos^{-1}(0.8850) \approx 207.7^\circ$$

\therefore Dino is 4.84 km from the start at a bearing of 208° .

$$\begin{aligned} 5 \quad a & \quad \overrightarrow{AB} - \overrightarrow{CB} \\ &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= \overrightarrow{AC} \end{aligned}$$

$$\begin{aligned} b & \quad \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} \\ &= \overrightarrow{AC} + \overrightarrow{CD} \\ &= \overrightarrow{AD} \end{aligned}$$

$$6 \quad a \quad \text{If } \overrightarrow{AB} = \frac{1}{2}\overrightarrow{CD} \text{ then}$$

$[AB] \parallel [CD]$ and $AB = \frac{1}{2}(CD)$

$$b \quad \text{If } \overrightarrow{AB} = 2\overrightarrow{AC} \text{ then}$$

$[AB] \parallel [AC]$ and $AB = 2(AC)$

$\therefore A, B$ and C are collinear and

$$AB = 2(AC).$$

So, C is the midpoint of $[AB]$.

$$7 \quad a \quad \mathbf{p} + \mathbf{r} - \mathbf{q} = \mathbf{0}$$

$$\therefore \mathbf{p} + \mathbf{r} = \mathbf{q}$$

$$b \quad \mathbf{l} + \mathbf{m} - \mathbf{n} + \mathbf{j} - \mathbf{k} = \mathbf{0}$$

$$\therefore \mathbf{l} + \mathbf{m} + \mathbf{j} = \mathbf{n} + \mathbf{k}$$

$$8 \quad a \quad \overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{r} + \mathbf{q}$$

$$b \quad \overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OR} + \overrightarrow{RQ} = -\mathbf{p} + \mathbf{r} + \mathbf{q}$$

$$c \quad \overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \mathbf{r} + \frac{1}{2}\mathbf{q}$$

$$d \quad \overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QN}$$

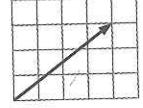
$$= \frac{1}{2}\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR}$$

$$= \frac{1}{2}(-\mathbf{p} + \mathbf{r} + \mathbf{q}) + \frac{1}{2}(-\mathbf{q})$$

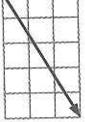
$$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{q}$$

$$= \frac{1}{2}\mathbf{r} - \frac{1}{2}\mathbf{p}$$

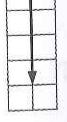
$$9 \quad a \quad \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$



$$b \quad \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$



$$c \quad \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$



$$10 \quad a \quad 2\mathbf{p} + \mathbf{q}$$

$$= 2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

$$b \quad \mathbf{q} - 3\mathbf{r}$$

$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -13 \end{pmatrix}$$

$$c \quad \mathbf{p} - \mathbf{q} + \mathbf{r}$$

$$= \begin{pmatrix} -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

$$11 \quad \overrightarrow{SP}$$

$$= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}$$

$$= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}$$

$$= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$12 \quad a \quad |\mathbf{r}| = \sqrt{4^2 + 1^2}$$

$$= \sqrt{17} \text{ units}$$

$$c \quad \mathbf{r} + \mathbf{s} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\therefore |\mathbf{r} + \mathbf{s}| = \sqrt{1^2 + 3^2}$$

$$= \sqrt{10} \text{ units}$$

$$b \quad |\mathbf{s}| = \sqrt{(-3)^2 + 2^2}$$

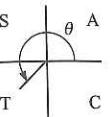
$$= \sqrt{13} \text{ units}$$

$$d \quad 2\mathbf{s} - \mathbf{r} = 2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 3 \end{pmatrix}$$

$$\therefore |2\mathbf{s} - \mathbf{r}| = \sqrt{(-10)^2 + 3^2}$$

$$= \sqrt{109} \text{ units}$$



$$13 \quad a \quad \overrightarrow{BC} = 2\overrightarrow{OA} = 2\mathbf{p}$$

Now $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BC}$

$$= -\mathbf{p} + \mathbf{q} + 2\mathbf{p}$$

$$= \mathbf{p} + \mathbf{q}$$

$$b \quad \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

$$= \mathbf{p} + \frac{1}{2}\overrightarrow{AC}$$

$$= \mathbf{p} + \frac{1}{2}(\mathbf{p} + \mathbf{q})$$

$$= \frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$

$$14 \quad a \quad \mathbf{p} - 3\mathbf{x} = \mathbf{0}$$

$$\therefore \mathbf{p} = 3\mathbf{x}$$

$$\therefore \frac{1}{3}\mathbf{p} = \mathbf{x}$$

$$\therefore \mathbf{x} = \begin{pmatrix} -1 \\ \frac{1}{3} \end{pmatrix}$$

$$b \quad 2\mathbf{q} - \mathbf{x} = \mathbf{r}$$

$$\therefore 2\mathbf{q} - \mathbf{r} = \mathbf{x}$$

$$\therefore \mathbf{x} = 2 \begin{pmatrix} 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$$

$$15 \quad \begin{array}{ccc} W & \xrightarrow{\hspace{2cm}} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\hspace{2cm}} & Z \end{array}$$

$$\overrightarrow{WY} = \begin{pmatrix} 3 & -3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$$\overrightarrow{XZ} = \begin{pmatrix} 4 & -2 \\ 10 & -5 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

So, $\overrightarrow{WY} = \overrightarrow{XZ}$
 $\therefore [WY]$ is parallel to $[XZ]$ and they are equal in length. This is sufficient to deduce that $WYZX$ is a parallelogram.

$$16 \quad r \begin{pmatrix} -2 \\ 1 \end{pmatrix} + s \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 13 \\ -24 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -2r + 3s \\ r - 4s \end{pmatrix} = \begin{pmatrix} 13 \\ -24 \end{pmatrix}$$

$$\therefore -2r + 3s = 13$$

$$r - 4s = -24 \quad \dots\dots (1)$$

$$\therefore -2r + 3s = 13$$

$$\frac{2r - 8s = -48}{\text{adding} \quad -5s = -35} \quad \{2 \times (1)\}$$

$$\therefore s = 7$$

$$\text{and in (1)} \quad r - 4(7) = -24$$

$$\therefore r = -24 + 28$$

$$\therefore r = 4 \quad \text{and} \quad s = 7$$

$$17 \quad \begin{array}{ccc} D & \xrightarrow{\hspace{2cm}} & B \\ \downarrow & & \downarrow \\ A & \xrightarrow{\hspace{2cm}} & C \end{array}$$

$$a \quad \overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB}$$

$$= \overrightarrow{OC} + \overrightarrow{OB}$$

$$= \mathbf{q} + \mathbf{r}$$

$$b \quad \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$

$$= \overrightarrow{OB} + \overrightarrow{OC}$$

$$= \mathbf{r} + \mathbf{q}$$

We see that $\overrightarrow{DB} = \overrightarrow{AC}$
 $\therefore [DB]$ is parallel to $[AC]$ and equal in length.

REVIEW SET 14B

$$1 \quad a \quad \overrightarrow{PQ} = \begin{pmatrix} -1 & -2 \\ 7 & -5 \\ 9 & -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 3 \end{pmatrix}$$

$$b \quad PQ = \sqrt{(-3)^2 + 12^2 + 3^2}$$

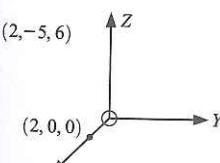
$$= \sqrt{162} \text{ units}$$

$$\therefore \text{distance}$$

$$= \sqrt{(2-2)^2 + (0-5)^2 + (0-6)^2}$$

$$= \sqrt{0+25+36}$$

$$= \sqrt{61} \text{ units}$$



$$2 \quad a \quad \mathbf{m} - \mathbf{n} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 11 \end{pmatrix}$$

$$b \quad 2\mathbf{n} - 3\mathbf{p} = 2 \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -8 \end{pmatrix} - \begin{pmatrix} -3 \\ 9 \\ 18 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -26 \end{pmatrix}$$

c $\mathbf{m} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}$ $\therefore |\mathbf{m} + \mathbf{p}| = \sqrt{25 + 0 + 49} = \sqrt{74}$ units

3 $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix}$

4 The vectors are parallel, so $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix} = k \begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$ $\therefore 3k = -12, km = -20, kn = 2$
 $\therefore k = -4, m = 5, n = -\frac{1}{2}$

5 $\overrightarrow{PQ} = \begin{pmatrix} 4 & -6 \\ 6 & -8 \\ 8 & -2 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ \therefore both \overrightarrow{PQ} and \overrightarrow{QR} are parallel to $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} 19 & -4 \\ 3 & -6 \\ 17 & -8 \end{pmatrix} = \begin{pmatrix} 15 \\ -3 \\ 9 \end{pmatrix} = 3 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ $\therefore [\text{PQ}] \parallel [\text{QR}]$ with Q common to both.
 $\therefore \text{P, Q, and R are collinear.}$
Now $\overrightarrow{PQ} : \overrightarrow{QR} = 2 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} : 3 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = 2 : 3$.

$\therefore \text{Q divides [PR] internally in the ratio } 2 : 3.$

6 As the vectors are perpendicular,

$$\begin{pmatrix} -4 \\ t+2 \\ t \end{pmatrix} \bullet \begin{pmatrix} t \\ 1+t \\ -3 \end{pmatrix} = 0$$

$$\therefore -4t + (t+2)(1+t) - 3t = 0$$

$$\therefore -4t + t^2 + 2t + 2t - 3t = 0$$

$$\therefore t^2 - 4t + 2 = 0$$

$$\therefore t = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2}$$

$$\therefore t = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

7 If θ is the angle then $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \sqrt{4 + 16 + 9} \sqrt{1 + 1 + 9} \cos \theta$
 $\therefore -2 - 4 + 9 = \sqrt{29} \sqrt{11} \cos \theta$
 $\therefore \frac{3}{\sqrt{29} \times \sqrt{11}} = \cos \theta \text{ and so } \theta \approx 80.3^\circ$

8 If D is the origin, (DA) the X-axis, (DC) the Y-axis and (DE) the Z-axis, then A is (4, 0, 0), C is (0, 8, 0) and G is (4, 8, 5).

$$\overrightarrow{AG} = \begin{pmatrix} 4 & -4 \\ 8 & 0 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 0 & -4 \\ 8 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$$

If the required angle is θ then $\begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} = \sqrt{0 + 64 + 25} \sqrt{16 + 64 + 0} \cos \theta$
 $\therefore 0 + 64 + 0 = \sqrt{89} \sqrt{80} \cos \theta$
 $\therefore \cos \theta = \frac{64}{\sqrt{89} \times \sqrt{80}}$ and so $\theta \approx 40.7^\circ$

9 a $\overrightarrow{PQ} = \begin{pmatrix} -4 & -2 \\ 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$

b $PQ = |\overrightarrow{PQ}| = \sqrt{36 + 1 + 9} = \sqrt{46}$ units

c The midpoint is at $\left(\frac{2+4}{2}, \frac{3+4}{2}, \frac{-1+2}{2}\right)$ which is $(-1, \frac{7}{2}, \frac{1}{2})$.

10 a $\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = -3 - 2 + 4 = -1$

b $\mathbf{p} + 2\mathbf{q} - \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$

c $\mathbf{p} \bullet \mathbf{r} = |\mathbf{p}| |\mathbf{r}| \cos \theta$ $\therefore \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \sqrt{1+4+1} \sqrt{1+1+4} \cos \theta$
 $\therefore -1 + 2 + 2 = \sqrt{6} \sqrt{6} \cos \theta$
 $\therefore 3 = 6 \cos \theta$

$$\therefore \cos \theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

11 $\overrightarrow{MK} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\overrightarrow{ML} = \begin{pmatrix} -2 & -4 \\ 1 & -1 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$$

$$\overrightarrow{LK} = \begin{pmatrix} 3 & -2 \\ 1 & -1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

$$\overrightarrow{LM} = \begin{pmatrix} 4 & -2 \\ 1 & -1 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

$\overrightarrow{MK} \bullet \overrightarrow{ML} = |\overrightarrow{MK}| |\overrightarrow{ML}| \cos M$

$\therefore 6 + 0 + 0 = \sqrt{1+0+1} \sqrt{36+0+0} \cos M$

$\therefore 6 = \sqrt{2} \times 6 \cos M$

$\therefore \cos M = \frac{1}{\sqrt{2}}$

$\therefore M = 45^\circ$

$\therefore \overrightarrow{LK} \bullet \overrightarrow{LM} = |\overrightarrow{LK}| |\overrightarrow{LM}| \cos L$

$\therefore 30 + 0 + 0 = \sqrt{25+0+1} \sqrt{36+0+0} \cos L$

$\therefore 30 = \sqrt{26} \times 6 \cos L$

$\therefore \frac{5}{\sqrt{26}} = \cos L$

$\therefore L \approx 11.3^\circ$

and $K \approx 180^\circ - 45^\circ - 11.3^\circ \approx 123.7^\circ$

12 If the angle is θ then $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \sqrt{9+1+4} \sqrt{4+25+1} \cos \theta$

$\therefore 6 + 5 - 2 = \sqrt{14} \sqrt{30} \cos \theta$

$\therefore \frac{9}{\sqrt{14} \times \sqrt{30}} = \cos \theta$

$\therefore \theta \approx 64.0^\circ$

13 $\overrightarrow{BA} = \begin{pmatrix} 4 & -1 \\ 2 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 3 & -1 \\ -3 & 5 \\ c-2 & 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ c-2 \end{pmatrix}$

But $\overrightarrow{BA} \bullet \overrightarrow{BC} = 0$

$\therefore 20 + 24 - 3(c-2) = 0$

$\therefore 44 = 3(c-2)$

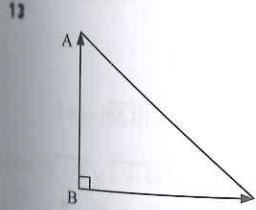
$\therefore 3c - 6 = 44$

$\therefore 3c = 50$

$\therefore c = \frac{50}{3}$

14 a $\mathbf{a} \bullet \mathbf{b}$ is a scalar, so in $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ we would be trying to find the scalar product of a scalar and a vector which is impossible.

b $\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}$ does not need a bracket about the $\mathbf{b} \times \mathbf{c}$ as this must be performed first; if we try to find $\mathbf{a} \bullet \mathbf{b}$ first we get a scalar cross a vector which is impossible.



15 a $\begin{pmatrix} \frac{4}{7} \\ \frac{1}{k} \end{pmatrix}$ is a unit vector if
 $\sqrt{\left(\frac{4}{7}\right)^2 + \left(\frac{1}{k}\right)^2} = 1$
 $\therefore \frac{16}{49} + \frac{1}{k^2} = 1$
 $\therefore \frac{1}{k^2} = \frac{33}{49}$
 $\therefore k = \pm \frac{7}{\sqrt{33}}$

b $\begin{pmatrix} k \\ k \end{pmatrix}$ is a unit vector if
 $\sqrt{k^2 + k^2} = 1$
 $\therefore 2k^2 = 1$
 $\therefore k^2 = \frac{1}{2}$
 $\therefore k = \pm \frac{1}{\sqrt{2}}$

REVIEW SET 14C

1 a $\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
 $= -3 + (-10)$
 $= -13$

b $\mathbf{p} - \mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix}$
 $= \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

c $\mathbf{q} \bullet (\mathbf{p} - \mathbf{r}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -6 \end{pmatrix}$
 $= -6 - 30$
 $= -36$

2 LHS = $\mathbf{p} \bullet (\mathbf{q} - \mathbf{r})$
 $= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \left[\begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right]$
 $= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 8 \end{pmatrix}$
 $= -9 - 16$
 $= -25$

RHS = $\mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$
 $= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix}$
 $= (-6 - 10) - (3 + 6)$
 $= -16 - 9$
 $= -25 \quad \therefore \text{LHS} = \text{RHS} \quad \checkmark$

3 Since they are perpendicular
 $\begin{pmatrix} 3 \\ 3 - 2t \end{pmatrix} \bullet \begin{pmatrix} t^2 + t \\ -2 \end{pmatrix} = 0$
 $\therefore 3(t^2 + t) - 2(3 - 2t) = 0$
 $\therefore 3t^2 + 3t - 6 + 4t = 0$
 $\therefore 3t^2 + 7t - 6 = 0$
 $\therefore (3t - 2)(t + 3) = 0$
 $\therefore t = \frac{2}{3} \text{ or } -3$

4 $\overrightarrow{AB} = \begin{pmatrix} -1 - 2 \\ 4 - 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
 $\overrightarrow{AC} = \begin{pmatrix} 3 - 2 \\ k - 3 \end{pmatrix} = \begin{pmatrix} 1 \\ k - 3 \end{pmatrix}$
Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$ {as $\widehat{BAC} = 90^\circ$ }
 $\therefore \begin{pmatrix} -3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ k - 3 \end{pmatrix} = 0$
 $\therefore -3 + k - 3 = 0$
 $\therefore k = 6$

5 One vector perpendicular to $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ as the dot product $= -20 + 20 = 0$
 \therefore all vectors have form $k \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $k \neq 0$.

6 $\overrightarrow{KL} = \begin{pmatrix} 3 - -2 \\ 2 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$
 $\overrightarrow{KM} = \begin{pmatrix} 1 - -2 \\ -3 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$
Now $\overrightarrow{KL} \bullet \overrightarrow{KM} = |\overrightarrow{KL}| |\overrightarrow{KM}| \cos K$
 $\therefore \begin{pmatrix} 5 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \sqrt{25+1} \sqrt{9+16} \cos K$
 $\therefore 15 - 4 = \sqrt{26} \sqrt{25} \cos K$
 $\therefore \cos K = \frac{11}{5\sqrt{26}}$
 $\therefore K \approx 64.4^\circ$

$\overrightarrow{LK} = -\overrightarrow{KL} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$
 $\overrightarrow{LM} = \begin{pmatrix} 1 - 3 \\ -3 - 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$
Now $\overrightarrow{LK} \bullet \overrightarrow{LM} = |\overrightarrow{LK}| |\overrightarrow{LM}| \cos L$
 $\therefore \begin{pmatrix} -5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -5 \end{pmatrix} = \sqrt{25+1} \sqrt{4+25} \cos L$
 $\therefore 10 + 5 = \sqrt{26} \sqrt{29} \cos L$
 $\therefore \cos L = \frac{15}{\sqrt{26} \times 29}$
 $\therefore L \approx 56.9^\circ$
 $\therefore M \approx 180^\circ - 56.89^\circ - 64.44^\circ \approx 58.7^\circ$

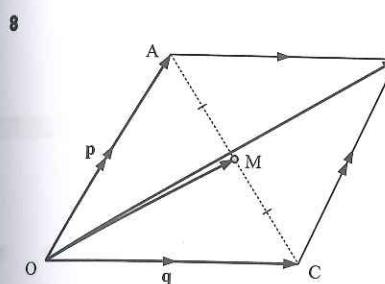
7 $4x - 5y = 11$ has gradient $\frac{4}{5} \therefore$ it has direction vector $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

$2x + 3y = 7$ has gradient $-\frac{2}{3} \therefore$ it has direction vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

If the angle is θ , $\begin{pmatrix} 5 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \sqrt{5^2 + 4^2} \sqrt{3^2 + (-2)^2} \cos \theta$

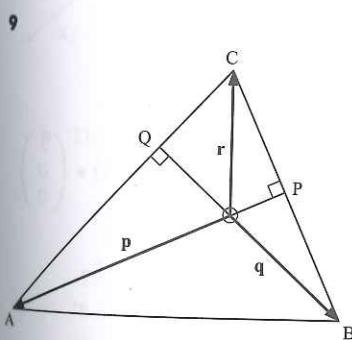
$$\therefore \frac{7}{\sqrt{41} \times 13} = \cos \theta$$

$\therefore \theta \approx 72.3^\circ \therefore$ the angle is 72.3° (or 107.7°)



a	i	\overrightarrow{OB}	ii	\overrightarrow{OM}
=	$\overrightarrow{OA} + \overrightarrow{AB}$	=	$\overrightarrow{OA} + \overrightarrow{AM}$	
=	$\overrightarrow{OA} + \overrightarrow{OC}$	=	$\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$	
=	$\mathbf{p} + \mathbf{q}$	=	$\mathbf{p} + \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OC})$	
=	$\mathbf{p} + \frac{1}{2}(-\mathbf{p} + \mathbf{q})$	=	$\mathbf{p} - \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$	
=	$\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$			

b We notice that $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$
 $\therefore [OM] \parallel [OB]$ and $OM = \frac{1}{2}(OB)$
So, O, M and B are collinear (as O is common) and hence M is the midpoint of [OB].



a	$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$	$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$
=	$-\mathbf{p} + \mathbf{r}$	= $-\mathbf{q} + \mathbf{r}$
=	$\mathbf{r} - \mathbf{p}$	= $\mathbf{r} - \mathbf{q}$

b	$[AP] \perp [BC]$ and $[BQ] \perp [AC]$
$\therefore \mathbf{p} \bullet \mathbf{r} - \mathbf{q} = 0$	$\therefore \mathbf{q} \bullet (\mathbf{r} - \mathbf{p}) = 0$
$\therefore \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} = 0$	$\therefore \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} = 0$
$\therefore \mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$	$\therefore \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$

c	$\mathbf{r} \bullet \overrightarrow{AB} = \mathbf{r} \bullet (-\mathbf{p} + \mathbf{q})$	$\therefore \mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q}$
=	= $-\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q}$	= $-\mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q}$ {from b}
=	= 0	and so $\mathbf{r} \perp \overrightarrow{AB} \therefore [OC] \perp [AB]$

10 a $2\mathbf{a} - 3\mathbf{b} = 2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \\ -12 \\ -7 \end{pmatrix}$

b $\mathbf{a} - 3\mathbf{x} = \mathbf{b} \therefore \mathbf{a} - \mathbf{b} = 3\mathbf{x} \therefore \mathbf{x} = \frac{1}{3}(\mathbf{a} - \mathbf{b}) = \frac{1}{3} \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{3} \\ -\frac{2}{3} \end{pmatrix}$

11 $|\mathbf{a}| = 3$, $|\mathbf{b}| = \sqrt{7}$ and $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

a $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$
 $\sqrt{1+4+9} = 3 \times \sqrt{7} \times \sin \theta$

$$\sin \theta = \frac{\sqrt{14}}{3\sqrt{7}} = \frac{\sqrt{2}}{3}$$

But $\cos^2 \theta = 1 - \sin^2 \theta$
 $\therefore \cos \theta = \pm \frac{\sqrt{7}}{3}$

Hence $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$
 $= 3 \times \sqrt{7} \times (\pm \frac{\sqrt{7}}{3})$
 $= \pm 7$

b Area $\Delta OAB = \frac{1}{2} |\mathbf{a} \times \mathbf{b}|$
 $= \frac{1}{2} \sqrt{14} \text{ units}^2$

c $V = \frac{1}{6} |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$

Now $\mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$$\therefore V = \frac{1}{6} \left| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \right|$$
 $= \frac{1}{6} |1 - 2 - 6|$
 $= \frac{7}{6} \text{ units}^3$

REVIEW SET 14D

1 $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{k}$

a $3\mathbf{a} - 2\mathbf{b} = 3(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - 2(\mathbf{i} - 2\mathbf{k})$
 $= 9\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} - 2\mathbf{i} + 4\mathbf{k}$
 $= 7\mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$

2 Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\therefore k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$

b $|\mathbf{a}| = \sqrt{3^2 + (-1)^2 + 2^2}$
 $= \sqrt{14} \text{ units}$

$\therefore |k\mathbf{a}| = \sqrt{(ka_1)^2 + (ka_2)^2 + (ka_3)^2}$
 $= \sqrt{k^2(a_1^2 + a_2^2 + a_3^2)}$
 $= \sqrt{k^2} \sqrt{a_1^2 + a_2^2 + a_3^2}$
 $= |k| |\mathbf{a}|$

3 a Given $P(-1, 2, 3)$ and $Q(4, 0, -1)$,

$$\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$$

b If the angle is α ,

$$|\overrightarrow{PQ}| \sqrt{1^2 + 0^2 + 0^2} \cos \alpha = \overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
 $\therefore \sqrt{25 + 4 + 16} \cos \alpha = 5$
 $\therefore \cos \alpha = \frac{5}{\sqrt{45}}$
 $\therefore \alpha \approx 41.8^\circ$

c 

$$\overrightarrow{QR} = 2\overrightarrow{RP}$$

$$\begin{pmatrix} a-4 \\ b \\ c+1 \end{pmatrix} = 2 \begin{pmatrix} -1-a \\ 2-b \\ 3-c \end{pmatrix}$$

$$a-4 = -2-2a, \quad b = 4-2b, \quad c+1 = 6-2c$$

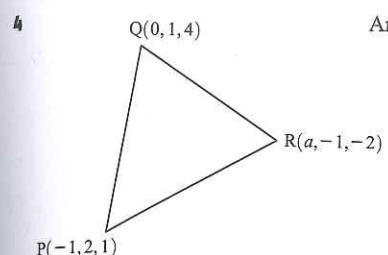
$$\therefore 3a = 2$$

$$\therefore a = \frac{2}{3}$$

$$\therefore b = \frac{4}{3}$$

$$\therefore c = \frac{5}{3}$$

$$\therefore R \text{ is at } (\frac{2}{3}, \frac{4}{3}, \frac{5}{3}).$$



$$\text{Area of } \Delta = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \left| \begin{pmatrix} 0 & -1 \\ 1 & -2 \\ 4 & -1 \end{pmatrix} \times \begin{pmatrix} a & -1 \\ -1 & -2 \\ -2 & -1 \end{pmatrix} \right|$$

$$= \frac{1}{2} \left| \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} a+1 \\ -3 \\ -3 \end{pmatrix} \right|$$

$$\therefore \frac{1}{2} \left| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ a+1 & -3 & -3 \end{matrix} \right| = \sqrt{118}$$

$$\therefore \left| \begin{matrix} -1 & 3 \\ -3 & a+1 \end{matrix} \mathbf{i} - \begin{matrix} 1 & 3 \\ a+1 & -3 \end{matrix} \mathbf{j} + \begin{matrix} 1 & -1 \\ a+1 & -3 \end{matrix} \mathbf{k} \right| = 2\sqrt{118}$$

$$|12\mathbf{i} - (-3-3a-3)\mathbf{j} + (-3+a+1)\mathbf{k}| = 2\sqrt{118}$$

$$\therefore \sqrt{144 + (3a+6)^2 + (a-2)^2} = 2\sqrt{118}$$

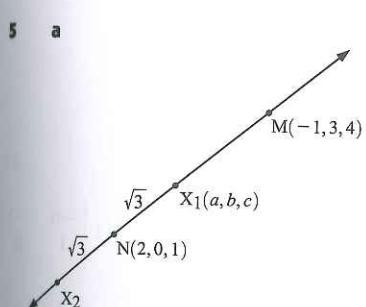
$$\therefore 144 + 9a^2 + 36a + 36 + a^2 - 4a + 4 = 472$$

$$\therefore 10a^2 + 32a - 288 = 0$$

$$\therefore 5a^2 + 16a - 144 = 0$$

$$\therefore (5a+36)(a-4) = 0$$

$$\therefore a = -\frac{36}{5} \text{ or } 4$$



$$\overrightarrow{MN} = \begin{pmatrix} 2 & -1 \\ 0 & -3 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{NX} = \begin{pmatrix} a-2 \\ b-0 \\ c-1 \end{pmatrix}, \quad |\overrightarrow{NX}| = \sqrt{3}, \quad \text{and} \quad \overrightarrow{NX} \parallel \overrightarrow{MN}$$

$$\therefore \begin{pmatrix} a-2 \\ b-0 \\ c-1 \end{pmatrix} = \pm \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \{ \text{as this vector has length } \sqrt{3} \}$$

$$\therefore a-2 = \pm 1, \quad b = \mp 1, \quad c-1 = \mp 1$$

$$\therefore a = 3 \text{ or } 1, \quad b = -1 \text{ or } 1, \quad c = 0 \text{ or } 2$$

$$\therefore X \text{ is at } (3, -1, 0) \text{ or } (1, 1, 2).$$

b The unit vector in the direction of \overrightarrow{MN} is $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

∴ the required vector is $\frac{2}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} \frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \end{pmatrix}$.

c $\overrightarrow{BC} = \begin{pmatrix} -1-2 \\ -9-0 \\ 4-1 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{CA} = \begin{pmatrix} 1-1 \\ -3-9 \\ 2-4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}$

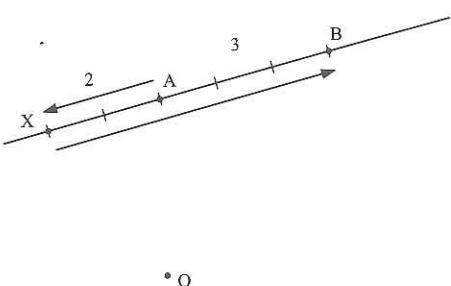
$$\therefore -\frac{1}{3}\overrightarrow{BC} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\overrightarrow{CA} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$\therefore -\frac{1}{3}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{CA} \quad \text{or} \quad \overrightarrow{BC} = -\frac{3}{2}\overrightarrow{CA}$$

$$\therefore [\overrightarrow{BC}] \parallel [\overrightarrow{CA}] \quad \text{so } B, C \text{ and } A \text{ are collinear.}$$

For C divides [BA] we need $\overrightarrow{BC} : \overrightarrow{CA} = -3 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} : 2 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = -3 : 2$

$$\therefore C \text{ divides [BA] externally in the ratio } 3 : 2.$$

7

Let X divide [AB] externally in the ratio 2 : 5.

$$\therefore \overrightarrow{AX} : \overrightarrow{XB} = -2 : 5$$

$$\text{Now } \overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$

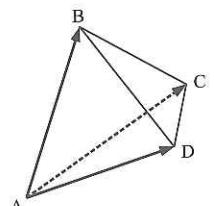
$$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{BA}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} -2 - 3 \\ 3 - 1 \\ 5 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} -5 \\ 4 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 - \frac{10}{3} \\ 3 + \frac{8}{3} \\ 5 + \frac{8}{3} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\frac{16}{3} \\ \frac{17}{3} \\ \frac{23}{3} \end{pmatrix}$$

$$\therefore X \text{ is } \left(-\frac{16}{3}, \frac{17}{3}, \frac{23}{3} \right).$$

8

Volume

$$= \frac{1}{6} |\overrightarrow{AB} \bullet (\overrightarrow{AC} \times \overrightarrow{AD})|$$

$$= \frac{1}{6} \begin{vmatrix} -4 & 1 & -1 \\ -5 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} \text{ units}^3$$

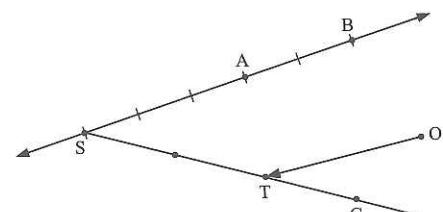
$$= \frac{1}{6} \begin{vmatrix} -4 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & 1 & -3 \end{vmatrix} - \begin{vmatrix} -5 & 1 & -1 \\ 1 & -3 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \frac{1}{6} |-4(1) - 1(14) - 1(-9)|$$

$$= \frac{1}{6} |-9|$$

$$= 1\frac{1}{2} \text{ units}^3$$

$$9. \overrightarrow{AS} : \overrightarrow{SB} = -3 : 5 \quad \text{and} \quad \overrightarrow{CT} : \overrightarrow{TS} = 1 : 2$$



$$\overrightarrow{OT} = \overrightarrow{OC} + \overrightarrow{CT}$$

$$= \mathbf{c} + \frac{1}{3}\overrightarrow{CS}$$

$$= \mathbf{c} + \frac{1}{3}(\overrightarrow{CB} + \overrightarrow{BS})$$

$$= \mathbf{c} + \frac{1}{3}(\overrightarrow{CO} + \overrightarrow{OB}) + \frac{1}{3} \times \frac{5}{2}\overrightarrow{BA}$$

$$= \mathbf{c} + \frac{1}{3}(-\mathbf{c} + \mathbf{b}) + \frac{5}{6}(\overrightarrow{BO} + \overrightarrow{OA})$$

$$= \mathbf{c} - \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{b} + \frac{5}{6}(-\mathbf{b} + \mathbf{a})$$

$$= \mathbf{c} - \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{b} - \frac{5}{6}\mathbf{b} + \frac{5}{6}\mathbf{a}$$

$$= \frac{5}{6}\mathbf{a} - \frac{1}{2}\mathbf{b} + \frac{2}{3}\mathbf{c}$$

$$10. \begin{pmatrix} 1 \\ r \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 0 \quad \therefore 2 + 2r - 2 = 0$$

$$\therefore 2r = 0$$

$$\therefore r = 0$$

So, we want a unit vector parallel to $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

Now $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ has length $\sqrt{1+0+4} = \sqrt{5}$

\therefore the vectors are $\pm \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, which are $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$ or $-\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$.

$$11. |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$$

$$\text{But } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \sqrt{1+9+16} = 3 \times 5 \times \sin \theta$$

$$\therefore \cos^2 \theta + \frac{26}{225} = 1$$

$$\therefore \frac{\sqrt{26}}{15} = \sin \theta$$

$$\therefore \cos^2 \theta = \frac{199}{225}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{199}}{15}$$

So, if θ is acute, $\cos \theta = \frac{\sqrt{149}}{15}$ and $\mathbf{u} \bullet \mathbf{v} = |\mathbf{a}| |\mathbf{b}| \cos \theta = 3 \times 5 \times \frac{\sqrt{149}}{15} = \sqrt{199}$

and if θ is obtuse, $\cos \theta = -\frac{\sqrt{149}}{15}$ and $\mathbf{u} \bullet \mathbf{v} = -\sqrt{199}$.

12. 3 vectors are coplanar if the volume of the tetrahedron defined by them is 0.

$$\therefore \begin{vmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ 1 & 2-t & t+1 \end{vmatrix} = 0 \quad \therefore 1 \begin{vmatrix} 2 & 3 \\ 2-t & t+1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 1 & t+1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ 1 & 2-t \end{vmatrix} = 0$$

$$\therefore 1(2t+2-6+3t) - 2(2t+2-3) - 3(4-2t-2) = 0$$

$$\therefore 5t-4-4t+2-6+6t=0$$

$$\therefore 7t=8$$

$$\therefore t=\frac{8}{7}$$

Placing a set of axes with origin at A, as shown, gives Q(4, 0, 7), M(0, 5, 7), D(0, 10, 0).

$$\overrightarrow{DQ} = \begin{pmatrix} 4-0 \\ 0-10 \\ 7-0 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ 7 \end{pmatrix}$$

$$\overrightarrow{DM} = \begin{pmatrix} 0-0 \\ 5-10 \\ 7-0 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}$$

$$\overrightarrow{DQ} \bullet \overrightarrow{DM} = |\overrightarrow{DQ}| |\overrightarrow{DM}| \cos \theta$$

$$\therefore 0+50+49 = \sqrt{16+100+49} \sqrt{0+25+49} \cos \theta$$

$$\therefore 99 = \sqrt{165} \sqrt{74} \cos \theta$$

$$\therefore \cos \theta = \frac{99}{\sqrt{165} \times \sqrt{74}}$$

$$\therefore \theta \approx 26.4^\circ$$

REVIEW SET 14E

$$1. \overrightarrow{AB} = \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ -7 \end{pmatrix}, \quad \overrightarrow{BC} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$$

$$\therefore AB = \sqrt{6^2 + 1^2 + (-4)^2} = \sqrt{53} \text{ units}$$

$$AC = \sqrt{0^2 + 2^2 + (-7)^2} = \sqrt{53} \text{ units}$$

$$BC = \sqrt{(-6)^2 + 1^2 + (-3)^2} = \sqrt{46} \text{ units}$$

$\therefore AB = AC$, so ABC is an isosceles triangle.

11 a $\vec{AB} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{k} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

\therefore the vectors $n(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$, $n \neq 0$ are perpendicular to the plane.

However, we require the vector to have length 10.

$$\therefore \sqrt{(5n)^2 + (4n)^2 + (3n)^2} = 10$$

$$\therefore 25n^2 + 16n^2 + 9n^2 = 100$$

$$\therefore 50n^2 = 100$$

$$\therefore n^2 = 2$$

$$\therefore n = \pm\sqrt{2}$$

\therefore the possible vectors are $\pm\sqrt{2} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$.

b Area of triangle ABC = $\frac{1}{2} |5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}|$

$$= \frac{1}{2} \sqrt{5^2 + 4^2 + 3^2}$$

$$= \frac{1}{2} \sqrt{50}$$

$$= \frac{5}{2}\sqrt{2} \text{ units}^2$$

c $\mathbf{b} - \mathbf{a} = \vec{AB} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ $\mathbf{c} - \mathbf{a} = \vec{AC} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\mathbf{d} - \mathbf{a} = \vec{AD} = \begin{pmatrix} -1 \\ 2 \\ k-1 \end{pmatrix}$

$$\therefore (\mathbf{b} - \mathbf{a}) \bullet (\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})$$

$$= \begin{vmatrix} -2 & 1 & 2 \\ 1 & -2 & 1 \\ -1 & 2 & k-1 \end{vmatrix} = -2 \begin{vmatrix} -2 & 1 \\ 2 & k-1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ -1 & k-1 \end{vmatrix} + 2 \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix}$$

$$= -2(-2k+2-2) - (k-1+1) + 2(2-2)$$

$$= 4k - k$$

$$= 3k$$

\therefore A, B, C and D are coplanar when $3k = 0 \therefore k = 0$

12 a $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix} \bullet \begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix} = 0$ b $\vec{KL} = \begin{pmatrix} -7 \\ 1 \\ 3 \end{pmatrix}$, $\vec{KM} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$, $\vec{LM} = \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$

$$\therefore (2-t)t + 12 + t(t+1) = 0$$

$$\therefore 2t - t^2 + 12 + t^2 + t = 0$$

$$\therefore 3t + 12 = 0$$

$$\therefore t = -4$$

Now $\vec{KM} \bullet \vec{LM} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$

$$= -2(5) - 2(-3) - 1(-4)$$

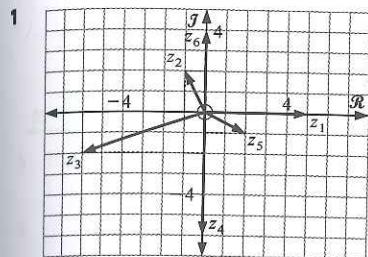
$$= 0$$

\therefore [KM] and [LM] are perpendicular
triangle KLM is right angled at M.

Chapter 15

COMPLEX NUMBERS

EXERCISE 15A.1



2 a

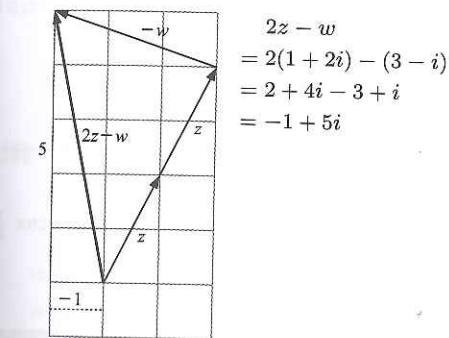
$$z + w = (1 + 2i) + (3 - i)$$

$$= 4 + i$$

b

$$z - w = (1 + 2i) - (3 - i)$$

$$= -2 + 3i$$



$$2z - w = 2(1 + 2i) - (3 - i)$$

$$= 2 + 4i - 3 + i$$

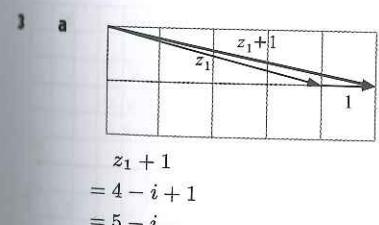
$$= -1 + 5i$$

d

$$w - 3z = (3 - i) - 3(1 + 2i)$$

$$= 3 - i - 3 - 6i$$

$$= -7i$$



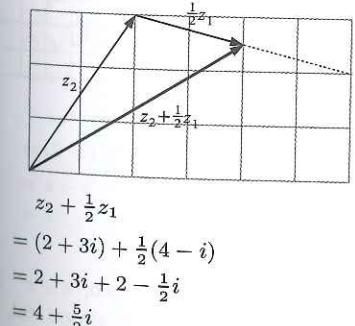
$$z_1 + 1 = 4 - i + 1$$

$$= 5 - i$$

b

$$z_1 + 2i = 4 - i + 2i$$

$$= 4 + i$$



$$z_2 + \frac{1}{2}z_1 = (2 + 3i) + \frac{1}{2}(4 - i)$$

$$= 2 + 3i + 2 - \frac{1}{2}i$$

$$= 4 + \frac{5}{2}i$$

d

$$\frac{z_1 + 4}{2} = \frac{4 - i + 4}{2}$$

$$= 4 - \frac{1}{2}i$$