

c $f''(x) = e^x$
 $\therefore f''(x) > 0$ for all x

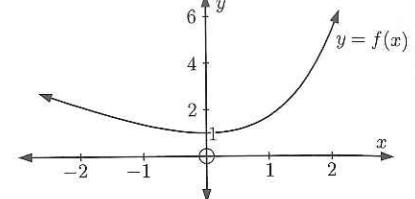
$\begin{array}{c} + \\ \xleftarrow{\hspace{-1cm}} \xrightarrow{\hspace{-1cm}} x \end{array}$
 $\therefore f(x)$ is concave up for all x

e Since a local minimum exists at $(0, 1)$,
 $f(x) \geq 1$ for all x

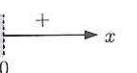
$\therefore e^x - x \geq 1$
 $\therefore e^x \geq x + 1$ for all x

5 a $f(x) = \ln(e^x + 3)$

$\therefore f'(x) = \frac{e^x}{e^x + 3}$

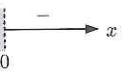


- 6 a $f(x) = x + \ln x$ is defined when $x > 0$
b $f'(x) = 1 + \frac{1}{x} = \frac{x+1}{x}$ which has sign diagram:

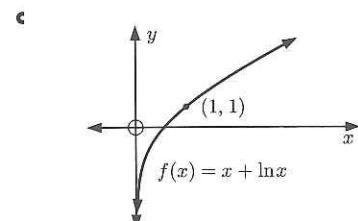


$\therefore f(x)$ is increasing for all $x > 0$.

$f''(x) = -\frac{1}{x^2}$ which has sign diagram:



$\therefore f(x)$ is concave down for all $x > 0$.

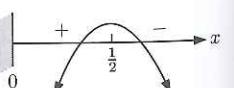


d $f(1) = 1 + \ln(1) = 1$
 $\therefore (1, 1)$ is the point of contact.
 $f'(1) = \frac{1+1}{1} = 2$
 \therefore the tangent at $x = 1$ has gradient 2,
so the normal has gradient $-\frac{1}{2}$
 \therefore the normal has equation $\frac{y-1}{x-1} = -\frac{1}{2}$
 $\therefore 2y-2 = -x+1$
 $\therefore x+2y=3$

- 7 Let the coordinates of B be $(x, 0)$, so the coordinates of A are (x, e^{-2x}) .

\therefore the area OBAC is $A = xe^{-2x}$

$\therefore \frac{dA}{dx} = (1)e^{-2x} + x(-2e^{-2x})$ {product rule}
 $= e^{-2x}(1-2x)$
 $= \frac{1-2x}{e^{2x}}$ and has sign diagram:



So, the maximum area occurs when $x = \frac{1}{2}$ and $y = e^{-2(\frac{1}{2})} = e^{-1} = \frac{1}{e}$

\therefore the coordinates of A are $(\frac{1}{2}, \frac{1}{e})$.

Chapter 20

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

EXERCISE 20A

1 a $y = \sin(2x)$

$\therefore \frac{dy}{dx} = \cos(2x) \frac{d}{dx}(2x)$
 $= 2\cos(2x)$

c $y = \cos(3x) - \sin x$
 $\therefore \frac{dy}{dx} = -\sin(3x) \times 3 - \cos x$
 $= -3\sin(3x) - \cos x$

e $y = \cos(3-2x)$
 $\therefore \frac{dy}{dx} = -\sin(3-2x) \times -2$
 $= 2\sin(3-2x)$

g $y = \sin\left(\frac{x}{2}\right) - 3\cos x$
 $\therefore \frac{dy}{dx} = \frac{1}{2}\cos\left(\frac{x}{2}\right) + 3\sin x$

i $y = 4\sin x - \cos(2x)$
 $\therefore \frac{dy}{dx} = 4\cos x + \sin(2x) \times 2$
 $= 4\cos x + 2\sin(2x)$

2 a $y = x^2 + \cos x$

$\therefore \frac{dy}{dx} = 2x - \sin x$

c $y = e^x \cos x$
 $\therefore \frac{dy}{dx} = e^x \cos x + e^x(-\sin x)$
 $= e^x \cos x - e^x \sin x$

e $y = \ln(\sin x)$
 $\therefore \frac{dy}{dx} = \frac{\cos x}{\sin x}$

g $y = \sin(3x)$
 $\therefore \frac{dy}{dx} = 3\cos(3x)$

b $y = \sin x + \cos x$

$\therefore \frac{dy}{dx} = \cos x - \sin x$

d $y = \sin(x+1)$
 $\therefore \frac{dy}{dx} = \cos(x+1) \frac{d}{dx}(x+1)$
 $= 1\cos(x+1)$
 $= \cos(x+1)$

f $y = \tan(5x)$
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2(5x)} \times 5$
 $= \frac{5}{\cos^2(5x)}$

h $y = 3\tan(\pi x)$
 $\therefore \frac{dy}{dx} = 3 \times \frac{1}{\cos^2(\pi x)} \times \pi$
 $= \frac{3\pi}{\cos^2(\pi x)}$

b $y = \tan x - 3\sin x$
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} - 3\cos x$

d $y = e^{-x} \sin x$
 $\therefore \frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$

f $y = e^{2x} \tan x$
 $\therefore \frac{dy}{dx} = 2e^{2x} \tan x + \frac{e^{2x}}{\cos^2 x}$

g $y = \cos(\frac{x}{2})$
 $\therefore \frac{dy}{dx} = -\frac{1}{2} \sin(\frac{x}{2})$

Teacher!

i $y = 3 \tan(2x)$
 $\therefore \frac{dy}{dx} = \frac{3}{\cos^2(2x)} \times 2$
 $= \frac{6}{\cos^2(2x)}$

k $y = \frac{\sin x}{x}$
 $\therefore \frac{dy}{dx} = \frac{(\cos x)(x) - \sin x \times 1}{x^2}$
 $= \frac{x \cos x - \sin x}{x^2}$

3 a $y = \sin(x^2)$
 $\therefore \frac{dy}{dx} = 2x \cos(x^2)$

c $y = \sqrt{\cos x} = (\cos x)^{\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = \frac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x)$
 $= -\frac{\sin x}{2\sqrt{\cos x}}$

e $y = \cos^3 x = (\cos x)^3$
 $\therefore \frac{dy}{dx} = 3 \cos^2 x \times (-\sin x)$
 $= -3 \sin x \cos^2 x$

g $y = \cos(\cos x)$
 $\therefore \frac{dy}{dx} = -\sin(\cos x) \times (-\sin x)$
 $= \sin x \sin(\cos x)$

i $y = \frac{1}{\sin x} = (\sin x)^{-1}$
 $\therefore \frac{dy}{dx} = -1(\sin x)^{-2} \times \cos x$
 $= -\frac{\cos x}{\sin^2 x}$

k $y = \frac{2}{\sin^2(2x)} = 2(\sin(2x))^{-2}$
 $\therefore \frac{dy}{dx} = -4(\sin(2x))^{-3} \times 2 \cos(2x)$
 $= -\frac{8 \cos(2x)}{\sin^3(2x)}$

4 a $f(x) = 2 \sin^3 x - 3 \sin x$
 $= 2(\sin x)^3 - 3 \sin x$
 $\therefore f'(x) = 2 \times 3(\sin x)^2 \times (\cos x) - 3 \cos x$
 $= -3 \cos x(1 - 2 \sin^2 x)$
 $= -3 \cos x \cos 2x$

j $y = x \cos x$
 $\therefore \frac{dy}{dx} = 1 \times \cos x + x(-\sin x)$
 $= \cos x - x \sin x$

l $y = x \tan x$
 $\therefore \frac{dy}{dx} = 1 \times \tan x + x \times \frac{1}{\cos^2 x}$
 $= \tan x + \frac{x}{\cos^2 x}$

b $y = \cos(\sqrt{x}) = \cos(x^{\frac{1}{2}})$
 $\therefore \frac{dy}{dx} = -\sin(x^{\frac{1}{2}}) \times \frac{1}{2}x^{-\frac{1}{2}}$
 $= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x})$

d $y = \sin^2 x = (\sin x)^2$
 $\therefore \frac{dy}{dx} = 2 \sin x \cos x$

f $y = \cos x \sin(2x)$
 $\therefore \frac{dy}{dx} = (-\sin x) \sin(2x) + \cos x(2 \cos(2x))$
 $= -\sin x \sin(2x) + 2 \cos x \cos(2x)$

h $y = \cos^3(4x) = (\cos(4x))^3$
 $\therefore \frac{dy}{dx} = 3(\cos(4x))^2 \times (-4 \sin(4x))$
 $= -12 \sin(4x) \cos^2(4x)$

j $y = \frac{1}{\cos(2x)} = (\cos(2x))^{-1}$
 $\therefore \frac{dy}{dx} = -1(\cos(2x))^{-2} \times (-2 \sin(2x))$
 $= \frac{2 \sin(2x)}{\cos^2(2x)}$

l $y = \frac{8}{\tan^3(\frac{x}{2})} = 8 [\tan(\frac{x}{2})]^{-3}$
 $\therefore \frac{dy}{dx} = -24 [\tan(\frac{x}{2})]^{-4} \times \frac{1}{2} \times \frac{1}{\cos^2(\frac{x}{2})}$
 $= \frac{-12}{\cos^2(\frac{x}{2}) \tan^4(\frac{x}{2})}$

b $f''(x) = -3(-\sin x \times \cos 2x)$
 $+ \cos x \times (-2) \sin 2x$
 $= 3 \sin x \cos 2x + 6 \cos x \sin 2x$

5 a If $y = \sin(2x + 3)$, then $\frac{dy}{dx} = 2 \cos(2x + 3)$ and $\frac{d^2y}{dx^2} = -4 \sin(2x + 3)$
 $\therefore \frac{d^2y}{dx^2} + 4y = -4 \sin(2x + 3) + 4 \sin(2x + 3) = 0$

b If $y = 2 \sin x + 3 \cos x$, then $y' = 2 \cos x - 3 \sin x$ and $y'' = -2 \sin x - 3 \cos x$
 $\therefore y'' + y = -2 \sin x - 3 \cos x + 2 \sin x + 3 \cos x = 0$

c $y = \frac{\cos x}{1 + \sin x}$
 $\therefore \frac{dy}{dx} = \frac{(-\sin x)(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2}$
 $= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$
 $= \frac{-1 - \sin x}{(1 + \sin x)^2}$
 $= \frac{-1}{1 + \sin x}$

Since $\frac{-1}{1 + \sin x}$ never equals 0, there are no horizontal tangents.

6 a $y = \sin x \quad \therefore \frac{dy}{dx} = \cos x$

When $x = 0$, $\frac{dy}{dx} = \cos 0 = 1$

\therefore the tangent has equation $\frac{y - 0}{x - 0} = 1$
 $\text{or } y = x$

c $y = \cos x \quad \therefore \frac{dy}{dx} = -\sin x$

When $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$
 $\text{and } \frac{dy}{dx} = -\sin \frac{\pi}{6} = -\frac{1}{2}$

So, the normal has gradient 2,

and its equation is $\frac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2$

$\therefore y - \frac{\sqrt{3}}{2} = 2x - \frac{\pi}{3}$
 $\therefore 2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

7 $d = 9.3 + 6.8 \cos(0.507t) \text{ m}$

$\therefore \frac{dd}{dt} = -6.8 \sin(0.507t) \times 0.507$
 $= -3.4476 \sin(0.507t)$

8 a $V(t) = 340 \sin(100\pi t)$

$\therefore V'(t) = 340 \cos(100\pi t) \times 100\pi$
 $= 34000\pi \cos(100\pi t)$

When $t = 0.01$,

$V'(0.01) = 34000\pi \times \cos \pi$
 $= -34000\pi \text{ units per second}$

b $y = \tan x \quad \therefore \frac{dy}{dx} = \frac{1}{\cos^2 x}$

When $x = 0$, $\frac{dy}{dx} = \frac{1}{\cos^2 0} = 1$

\therefore the tangent has equation $\frac{y - 0}{x - 0} = 1$
 $\text{or } y = x$

d $y = \frac{1}{\sin(2x)} = (\sin(2x))^{-1}$

$\therefore \frac{dy}{dx} = -1(\sin(2x))^{-2} \times 2 \cos(2x)$
 $= -\frac{2 \cos(2x)}{(\sin(2x))^2}$

When $x = \frac{\pi}{4}$, $y = 1$
 $\text{and } \frac{dy}{dx} = -\frac{2 \cos \frac{\pi}{2}}{(\sin \frac{\pi}{2})^2} = 0$

\therefore the gradient of the normal is undefined,
so the normal is $x = \frac{\pi}{4}$.

a When $t = 8$, $\frac{dd}{dt} \approx 2.731 > 0$

\therefore the tide is rising.

b When $t = 8$, the tide is rising at the rate of 2.73 m per hour.

b When $V(t)$ is a maximum,
 $V'(t)$ must be 0 units per second.

- 9** a The distance from $A(-x, 0)$ to $P(\cos t, \sin t)$ is fixed at 2 m.

$$\cos t = \frac{OQ}{1} = OQ$$

$$\therefore (\cos t + x)^2 + \sin^2 t = 2^2 \quad \{\text{Pythagoras in triangle APQ}\}$$

$$\therefore (\cos t + x)^2 = 4 - \sin^2 t$$

$$\therefore x + \cos t = \pm \sqrt{4 - \sin^2 t}$$

$$\therefore \text{since } x > 0, \quad x = \sqrt{4 - \sin^2 t} - \cos t$$

$$\begin{aligned} \text{b} \quad \text{Now } \frac{dx}{dt} &= \frac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2 \sin t \cos t) + \sin t \\ &= \frac{-\sin t \cos t}{\sqrt{4 - \sin^2 t}} + \sin t \end{aligned}$$

i When $t = 0$,

$$\sin t = 0 \text{ and } \cos t = 1$$

$$\therefore \frac{dx}{dt} = 0 + 0$$

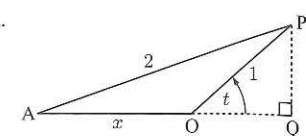
$$= 0 \text{ ms}^{-1}$$

ii When $t = \frac{\pi}{2}$,

$$\sin t = 1 \text{ and } \cos t = 0$$

$$\therefore \frac{dx}{dt} = 0 + \sin \frac{\pi}{2}$$

$$= 1 \text{ ms}^{-1}$$



iii When $t = \frac{2\pi}{3}$,

$$\sin t = \frac{\sqrt{3}}{2} \text{ and } \cos t = -\frac{1}{2}$$

$$\therefore \frac{dx}{dt} = \frac{-\frac{\sqrt{3}}{2}(-\frac{1}{2})}{\sqrt{4 - \frac{3}{4}}} + \frac{\sqrt{3}}{2}$$

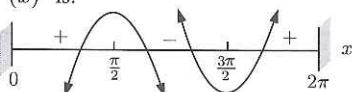
$$\approx 1.11 \text{ ms}^{-1}$$

- 10** a If $f(x) = \sin x$ then $f'(x) = \cos x$

Stationary points occur when $f'(x) = 0$,

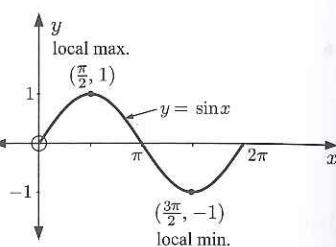
which is when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Sign diagram for $f'(x)$ is:



There is a local maximum at $(\frac{\pi}{2}, 1)$

and a local minimum at $(\frac{3\pi}{2}, -1)$.



- b If $f(x) = \cos(2x)$ then $f'(x) = -2 \sin(2x)$

$$\therefore f'(x) = 0 \text{ when } -2 \sin(2x) = 0$$

$$\therefore \sin(2x) = 0$$

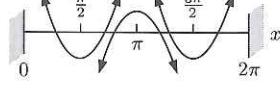
$$\therefore 2x = k\pi \text{ for any integer } k$$

$$\therefore x = \frac{k\pi}{2}$$

On the domain $0 \leq x \leq 2\pi$, $f'(x) = 0$

when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π .

Sign diagram for $f'(x)$ is:



There are local maxima at $(0, 1), (\pi, 1), (2\pi, 1)$ and local minima at $(\frac{\pi}{2}, -1), (\frac{3\pi}{2}, -1)$.

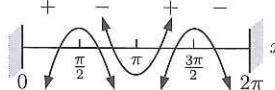
- c If $f(x) = \sin^2 x$ then $f'(x) = 2 \sin x \cos x = \sin(2x)$

$$\therefore f'(x) = 0 \text{ when } \sin(2x) = 0$$

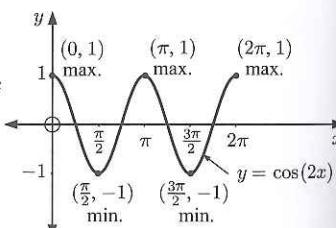
Using b, we know that on the domain $0 \leq x \leq 2\pi$

$$f'(x) = 0 \text{ when } x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ and } 2\pi.$$

Sign diagram for $f'(x)$ is:



There are local minima at $(0, 0), (\pi, 0), (2\pi, 0)$ and local maxima at $(\frac{\pi}{2}, 1), (\frac{3\pi}{2}, 1)$.



- d If $f(x) = e^{\sin x}$ then $f'(x) = e^{\sin x} \times \cos x$

$$\therefore f'(x) = 0 \text{ when } \cos x e^{\sin x} = 0$$

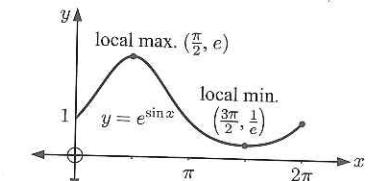
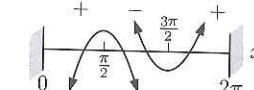
$$\therefore \cos x = 0 \quad \{e^{\sin x} > 0 \text{ for all } x\}$$

$$\therefore x = \frac{\pi}{2} + k\pi, \quad k \text{ an integer}$$

On the domain $0 \leq x \leq 2\pi$, $f'(x) = 0$

when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Sign diagram for $f'(x)$ is:



There is a local maximum at $(\frac{\pi}{2}, e)$

and a local minimum at $(\frac{3\pi}{2}, \frac{1}{e})$.

- e If $f(x) = \sin(2x) + 2 \cos x$ then $f'(x) = 2 \cos(2x) - 2 \sin x$

$$\therefore f'(x) = 0 \text{ when } 2 \cos(2x) - 2 \sin x = 0$$

$$\therefore 2(1 - 2 \sin^2 x) - 2 \sin x = 0$$

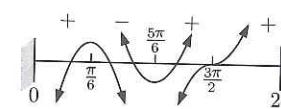
$$\therefore -2(2 \sin^2 x + \sin x - 1) = 0$$

$$\therefore -2(2 \sin x - 1)(\sin x + 1) = 0$$

$$\therefore \text{when } \sin x = \frac{1}{2} \text{ or } \sin x = -1$$

On the domain $0 \leq x \leq 2\pi$, when $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

Sign diagram of $f'(x)$:



$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) + 2 \cos\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} + 2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{6}\right) = \sin\left(\frac{10\pi}{6}\right) + 2 \cos\left(\frac{5\pi}{6}\right)$$

$$= -\frac{\sqrt{3}}{2} + 2 \times \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$$

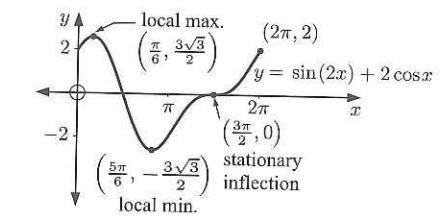
$$f\left(\frac{3\pi}{2}\right) = \sin(3\pi) + 2 \cos\left(\frac{3\pi}{2}\right)$$

$$= 0 + 2 \times 0 = 0$$

∴ there is a local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$,

a local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$

and a stationary point of inflection at $(\frac{3\pi}{2}, 0)$.



- 11 $x(t) = 1 - 2 \cos t$ cm

$$\therefore v(t) = x'(t) = 2 \sin t$$

$$\therefore a(t) = v'(t) = 2 \cos t$$

- a When $t = 0$,

$$x(0) = 1 - 2 \cos 0$$

$$= -1 \text{ cm}$$

$$v(0) = 2 \sin 0$$

$$= 0 \text{ cm s}^{-1}$$

$$a(0) = 2 \cos 0$$

$$= 2 \text{ cm s}^{-2}$$

- b When $t = \frac{\pi}{4}$,

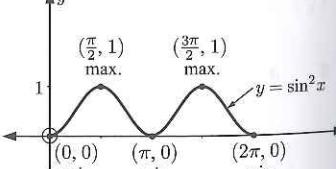
$$x\left(\frac{\pi}{4}\right) = 1 - \frac{2}{\sqrt{2}}$$

$$= 1 - \sqrt{2} \text{ cm}$$

$$v\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ cm s}^{-1}$$

$$a\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ cm s}^{-2}$$

The particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm s $^{-1}$ with increasing speed.



$$\begin{aligned}\therefore L &= BX + \frac{2BX}{AX} = \frac{2}{\sin \theta} + \frac{2\left(\frac{2}{\sin \theta}\right)}{\left(\frac{2 \cos \theta}{\sin \theta}\right)} \\ &= \frac{2}{\sin \theta} + 2\left(\frac{2}{\sin \theta}\right)\left(\frac{\sin \theta}{2 \cos \theta}\right) \\ &= \frac{2}{\cos \theta} + \frac{2}{\sin \theta} \quad \text{as required}\end{aligned}$$

b Now $L = \frac{2}{\cos \theta} + \frac{2}{\sin \theta} = 2(\cos \theta)^{-1} + 2(\sin \theta)^{-1}$

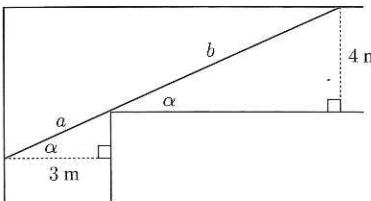
$$\begin{aligned}\therefore \frac{dL}{d\theta} &= -2(\cos \theta)^{-2} \times (-\sin \theta) - 2(\sin \theta)^{-2} \times \cos \theta \\ &= \frac{2 \sin \theta}{\cos^2 \theta} - \frac{2 \cos \theta}{\sin^2 \theta} \\ &= \frac{2 \sin^3 \theta - 2 \cos^3 \theta}{\sin^2 \theta \cos^2 \theta}\end{aligned}$$

c Now $\frac{dL}{d\theta} = 0$ when $2 \sin^3 \theta - 2 \cos^3 \theta = 0$

$$\begin{aligned}\therefore 2 \sin^3 \theta &= 2 \cos^3 \theta \\ \therefore \tan^3 \theta &= 1 \\ \therefore \tan \theta &= 1 \\ \therefore \text{since } 0 < \theta < 90^\circ, \quad \theta &= 45^\circ\end{aligned}$$

Sign diagram of $\frac{dL}{d\theta}$ is:

∴ the ladder is shortest when $\theta = 45^\circ$

$$\begin{aligned}\therefore \frac{1}{\cos \theta} &= \sqrt{2} \quad \text{and} \quad \frac{1}{\sin \theta} = \sqrt{2} \\ \therefore L_{\min} &= 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2} \text{ m}\end{aligned}$$
5

$$\begin{aligned}\cos \alpha &= \frac{3}{a} \quad \text{and} \quad \sin \alpha = \frac{4}{b} \\ \therefore a &= \frac{3}{\cos \alpha} \quad \text{and} \quad b = \frac{4}{\sin \alpha}\end{aligned}$$

Now $L = a + b$

$$\begin{aligned}\therefore L &= \frac{3}{\cos \alpha} + \frac{4}{\sin \alpha} \\ &= 3(\cos \alpha)^{-1} + 4(\sin \alpha)^{-1}\end{aligned}$$

$$\begin{aligned}\therefore \frac{dL}{d\alpha} &= -3(\cos \alpha)^{-2} \times (-\sin \alpha) - 4(\sin \alpha)^{-2} \times \cos \alpha \\ &= \frac{3 \sin \alpha}{\cos^2 \alpha} - \frac{4 \cos \alpha}{\sin^2 \alpha} \\ &= \frac{3 \sin^3 \alpha - 4 \cos^3 \alpha}{\cos^2 \alpha \sin^2 \alpha}\end{aligned}$$

Sign diagram of $\frac{dL}{d\alpha}$ is:

∴ L is minimised when $\alpha \approx 47.74^\circ$ and $L = \frac{3}{\cos \alpha} + \frac{4}{\sin \alpha} \approx 9.87 \text{ m}$

REVIEW SET 20

1 a $\frac{d}{dx} (\sin(5x) \ln x) = \frac{d}{dx} (\sin(5x)) \ln x + \sin(5x) \frac{d}{dx} (\ln x) \quad \{\text{product rule}\}$

$$= 5 \cos(5x) \ln x + \frac{\sin(5x)}{x}$$

b $\frac{d}{dx} (\sin x \cos(2x)) = \frac{d}{dx} (\sin x) \cos(2x) + \sin x \frac{d}{dx} (\cos(2x)) \quad \{\text{product rule}\}$

$$= \cos x \cos(2x) + \sin x (-2 \sin(2x))$$

$$= \cos x \cos(2x) - 2 \sin x \sin(2x)$$

c $\frac{d}{dx} (e^{-2x} \tan x) = \frac{d}{dx} (e^{-2x}) \tan x + e^{-2x} \frac{d}{dx} (\tan x) \quad \{\text{product rule}\}$

$$= -2e^{-2x} \tan x + \frac{e^{-2x}}{\cos^2 x}$$

d $\frac{d}{dx} (10x - \sin(10x)) = 10 - 10 \cos(10x)$

e $\frac{d}{dx} \left(\ln \left(\frac{1}{\cos x} \right) \right) = \frac{1}{\left(\frac{1}{\cos x} \right)} \times \frac{d}{dx} \left(\frac{1}{\cos x} \right) \quad \{\text{chain rule}\}$

$$= \cos x \times \frac{d}{dx} ((\cos x)^{-1})$$

$$= \cos x \times (-(\cos x)^{-2} \times (-\sin x))$$

$$= \frac{\cos x \sin x}{\cos^2 x} = \tan x$$

f $\frac{d}{dx} (\sin(5x) \ln(2x)) = \frac{d}{dx} (\sin(5x)) \ln(2x) + \sin(5x) \frac{d}{dx} (\ln(2x)) \quad \{\text{product rule}\}$

$$= 5 \cos(5x) \ln(2x) + \sin(5x) \times \frac{2}{2x}$$

$$= 5 \cos(5x) \ln(2x) + \frac{\sin(5x)}{x}$$

2 $y = x \tan x \quad \therefore \frac{dy}{dx} = 1 \times \tan x + x \times \left(\frac{1}{\cos^2 x} \right) = \tan x + \frac{x}{\cos^2 x}$

Now $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\tan \frac{\pi}{4} = 1$

∴ at $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$ and $\frac{dy}{dx} = 1 + \frac{\frac{\pi}{4}}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1 + \frac{\pi}{2}$

∴ the equation of the tangent is $\frac{y - \frac{\pi}{4}}{x - \frac{\pi}{4}} = 1 + \frac{\pi}{2}$

$$\begin{aligned}\therefore y - \frac{\pi}{4} &= (1 + \frac{\pi}{2})(x - \frac{\pi}{4}) \\ &= x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8}\end{aligned}$$

$$\begin{aligned}\therefore y &= (1 + \frac{\pi}{2})x - \frac{\pi^2}{8} \\ \therefore 2y &= (2 + \pi)x - \frac{\pi^2}{4}\end{aligned}$$

∴ $(2 + \pi)x - 2y = \frac{\pi^2}{4}$ as required

3 a $f(x) = 3 \sin x - 4 \cos(2x)$

∴ $f'(x) = 3 \cos x + 8 \sin(2x)$

∴ $f''(x) = -3 \sin x + 16 \cos(2x)$

b $f(x) = x^{\frac{1}{2}} \cos(4x)$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \cos(4x) + x^{\frac{1}{2}}(-4 \sin(4x)) \quad \{\text{product rule}\}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} \cos(4x) - 4x^{\frac{1}{2}} \sin(4x)$$

and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \cos(4x) + \frac{1}{2}x^{-\frac{1}{2}}(-4 \sin(4x)) - [2x^{-\frac{1}{2}} \sin(4x) + 4x^{\frac{1}{2}} \times 4 \cos(4x)]$

$$= -\frac{1}{4}x^{-\frac{3}{2}} \cos(4x) - 4x^{-\frac{1}{2}} \sin(4x) - 16x^{\frac{1}{2}} \cos(4x)$$

4 **a** $x(t) = 3 + \sin(2t)$ cm, $t \geq 0$ s $\therefore x(0) = 3$ cm

 $v(t) = x'(t) = 0 + 2 \cos(2t)$ cm s $^{-1}$ $v(0) = 2$ cm s $^{-1}$
 $a(t) = v'(t) = -4 \sin(2t)$ cm s $^{-2}$ $a(0) = 0$ cm s $^{-2}$

\therefore initially the particle is 3 cm right of O, moving right at a speed of 2 cm s $^{-1}$.

b $x'(t) = 0$ when $2 \cos(2t) = 0$

 $\therefore \cos(2t) = 0$
 $\therefore 2t = \frac{\pi}{2} + k\pi$

For the interval $0 \leq t \leq \pi$, $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$



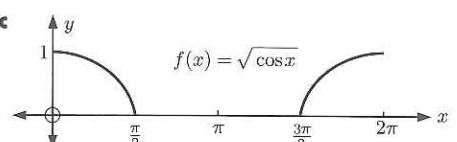
\therefore the particle reverses direction at $t = \frac{\pi}{4}, \frac{3\pi}{4}$

c $x(0) = 3$, $x\left(\frac{\pi}{4}\right) = 3 + \sin\left(\frac{\pi}{2}\right) = 4$,

 $x\left(\frac{3\pi}{4}\right) = 3 + \sin\left(\frac{3\pi}{2}\right) = 3 - 1 = 2$,
 $x(\pi) = 3 + \sin(2\pi) = 3$
 \therefore the total distance travelled = 1 + 2 + 1 = 4 cm.

5 **a** $f(x) = \sqrt{\cos x}$, $0 \leq x \leq 2\pi$

$f(x)$ is meaningful when $\cos x \geq 0$, which is when $0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$.



6 **a** $s(t) = 30 + \cos(\pi t)$ cm, $t \geq 0$

 $\therefore v(t) = s'(t) = -\pi \sin(\pi t)$

So, $v(0) = 0$ cm s $^{-1}$, $v\left(\frac{1}{2}\right) = -\pi$ cm s $^{-1}$, $v(1) = 0$ cm s $^{-1}$, $v\left(\frac{3}{2}\right) = \pi$ cm s $^{-1}$, $v(2) = 0$ cm s $^{-1}$

b The cork is falling when $v(t) \leq 0$, which is for $0 \leq t \leq 1$, $2 \leq t \leq 3$, ...

 \therefore the cork is falling for $2n \leq t \leq 2n+1$, $n \in \{0, 1, 2, 3, \dots\}$

b $f(x) = (\cos x)^{\frac{1}{2}}$

 $\therefore f'(x) = \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$
 $= \frac{-\sin x}{2\sqrt{\cos x}}$

$\therefore f'(x) = 0$ when $-\sin x = 0$

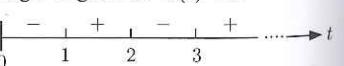
For $0 \leq x \leq 2\pi$, this is when $x = 0, \pi, 2\pi$.

Sign diagram for $f'(x)$ is:



$f(x)$ is increasing for $\frac{3\pi}{2} \leq x \leq 2\pi$ and decreasing for $0 \leq x \leq \frac{\pi}{2}$.

Sign diagram of $v(t)$ is:



7 **a** $\sin \theta = \frac{NA}{x} = \frac{1}{x}$

 $\therefore \frac{1}{x^2} = \sin^2 \theta$

\therefore at A, $I = \frac{\sqrt{8} \cos \theta}{x^2} = \sqrt{8} \cos \theta \sin^2 \theta$

b $\frac{dI}{d\theta} = \sqrt{8}(-\sin \theta) \sin^2 \theta + \sqrt{8} \cos \theta(2 \sin \theta \cos \theta)$
 $= \sqrt{8} \sin \theta[2 \cos^2 \theta - \sin^2 \theta]$
 $= \sqrt{8} \sin \theta[2(1 - \sin^2 \theta) - \sin^2 \theta]$
 $= \sqrt{8} \sin \theta[2 - 3 \sin^2 \theta]$

$\frac{dI}{d\theta} = 0$ when $\sin \theta = \sqrt{\frac{2}{3}}$, $0 < \theta < \frac{\pi}{2}$

and the sign diagram of $\frac{dI}{d\theta}$ is:

\therefore the maximum illumination at A is obtained when $\sin \theta = \sqrt{\frac{2}{3}}$.

$\therefore x = \frac{1}{\sin \theta} = \sqrt{\frac{3}{2}}$

$\therefore h = \sqrt{x^2 - NA^2} = \sqrt{\frac{3}{2} - 1} = \frac{1}{\sqrt{2}}$

\therefore the bulb is $\frac{1}{\sqrt{2}}$ m above the floor.

8 **a** $y = \frac{1}{\sin x} = (\sin x)^{-1}$

When $x = \frac{\pi}{3}$, $y = \frac{1}{\sin(\frac{\pi}{3})} = \frac{2}{\sqrt{3}}$

$\therefore \frac{dy}{dx} = -(\sin x)^{-2}(\cos x)$
 $= -\frac{\cos x}{\sin^2 x}$

and $\frac{dy}{dx} = -\frac{\cos(\frac{\pi}{3})}{\sin^2(\frac{\pi}{3})} = -\frac{\frac{1}{2}}{(\frac{\sqrt{3}}{2})^2} = -\frac{2}{3}$

\therefore the tangent has equation $\frac{y - \frac{2}{\sqrt{3}}}{x - \frac{\pi}{3}} = -\frac{2}{3}$ which is $3y - 2\sqrt{3} = -2x + \frac{2\pi}{3}$

or $2x + 3y = 2\sqrt{3} + \frac{2\pi}{3}$

b $y = \cos(\frac{x}{2})$

When $x = \frac{\pi}{2}$, $y = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\therefore \frac{dy}{dx} = -\frac{1}{2} \sin(\frac{x}{2})$

and $\frac{dy}{dx} = -\frac{1}{2} \sin(\frac{\pi}{4}) = -\frac{1}{2\sqrt{2}}$

\therefore the normal has gradient $2\sqrt{2}$, and its equation is $\frac{y - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{2}} = 2\sqrt{2}$

$\therefore y - \frac{1}{\sqrt{2}} = 2\sqrt{2}x - \pi\sqrt{2}$

$\therefore y - 2\sqrt{2}x = \frac{1}{\sqrt{2}} - \pi\sqrt{2}$

or $\sqrt{2}y - 4x = 1 - 2\pi$

9 **a** $f(x) = -10 \sin 2x \cos 2x$, $0 \leq x \leq \pi$

$\therefore f(x) = -5 \sin 4x$ { $2 \sin A \cos A = \sin 2A$ }

b $f'(x) = -20 \cos 4x$

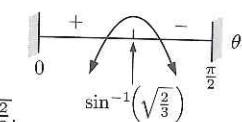
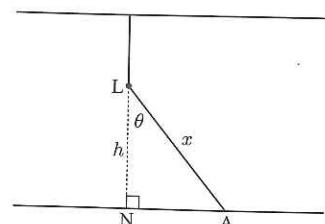
If $f'(x) = 0$, $-20 \cos 4x = 0$

$\therefore \cos 4x = 0$

$\therefore 4x = \frac{\pi}{2} + n\pi$, n any integer

$\therefore x = \frac{\pi}{8} + \frac{n\pi}{4}$

So, for the domain $0 \leq x \leq \pi$, $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$



$\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$