

Chapter 21

INTEGRATION

EXERCISE 21A

1 a i $\frac{d}{dx}(x^2) = 2x$
 $\therefore \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$
 \therefore the antiderivative of x is $\frac{1}{2}x^2$

iii $\frac{d}{dx}(x^6) = 6x^5$
 $\therefore \frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$
 \therefore the antiderivative of x^5 is $\frac{1}{6}x^6$

v $\frac{d}{dx}(x^{-3}) = -3x^{-4}$
 $\therefore \frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$
 \therefore the antiderivative of x^{-4} is $-\frac{1}{3}x^{-3}$

vii $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$
 $\therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = x^{-\frac{1}{2}}$
 \therefore the antiderivative of $x^{-\frac{1}{2}}$ is $2x^{\frac{1}{2}} = 2\sqrt{x}$

b the antiderivative of x^n is $\frac{x^{n+1}}{n+1}$

2 a i $\frac{d}{dx}(e^{2x}) = 2e^{2x}$
 $\therefore \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$
 \therefore the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$

iii $\frac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$
 $\therefore \frac{d}{dx}\left(2e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$
 \therefore the antiderivative of $e^{\frac{1}{2}x}$ is $2e^{\frac{1}{2}x}$

v $\frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$
 $\therefore \frac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$
 \therefore the antiderivative of $e^{\pi x}$ is $\frac{1}{\pi}e^{\pi x}$

b the antiderivative of e^{kx} is $\frac{1}{k}e^{kx}$

ii $\frac{d}{dx}(x^3) = 3x^2$
 $\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$
 \therefore the antiderivative of x^2 is $\frac{1}{3}x^3$

iv $\frac{d}{dx}(x^{-1}) = -x^{-2}$
 $\therefore \frac{d}{dx}(-x^{-1}) = x^{-2}$
 \therefore the antiderivative of x^{-2} is $-x^{-1}$ or $-\frac{1}{x}$

vi $\frac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}}$
 $\therefore \frac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$
 \therefore the antiderivative of $x^{\frac{1}{3}}$ is $\frac{3}{4}x^{\frac{4}{3}}$

ii $\frac{d}{dx}(e^{5x}) = 5e^{5x}$
 $\therefore \frac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$
 \therefore the antiderivative of e^{5x} is $\frac{1}{5}e^{5x}$

iv $\frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$
 $\therefore \frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$
 \therefore the antiderivative of $e^{0.01x}$ is $100e^{0.01x}$

vi $\frac{d}{dx}\left(e^{\frac{x}{3}}\right) = \frac{1}{3}e^{\frac{x}{3}}$
 $\therefore \frac{d}{dx}\left(3e^{\frac{x}{3}}\right) = e^{\frac{x}{3}}$
 \therefore the antiderivative of $e^{\frac{x}{3}}$ is $3e^{\frac{x}{3}}$

3 a $\frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$
 $\therefore \frac{d}{dx}(2x^3 + 2x^2) = 6x^2 + 4x$
 \therefore the antiderivative of $6x^2 + 4x$ is $2x^3 + 2x^2$

c $\frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$
 $= \frac{3}{2}\sqrt{x}$
 $\therefore \frac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$
 \therefore the antiderivative of \sqrt{x} is $\frac{2}{3}x\sqrt{x}$

b $\frac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$
 $\therefore \frac{d}{dx}\left(\frac{1}{3}e^{3x+1}\right) = e^{3x+1}$
 \therefore the antiderivative of e^{3x+1} is $\frac{1}{3}e^{3x+1}$

d $\frac{d}{dx}((2x+1)^4) = 4(2x+1)^3 \times 2$
 $= 8(2x+1)^3$
 $\therefore \frac{d}{dx}\left(\frac{1}{8}(2x+1)^4\right) = (2x+1)^3$
 \therefore the antiderivative of $(2x+1)^3$ is $\frac{1}{8}(2x+1)^4$

EXERCISE 21B

1 a $\int_a^a f(x) dx = F(a) - F(a) = 0$
 $\int_a^a f(x) dx$ = area of the strip between $x=a$ and $x=a$.

This strip has 0 width, so its area = 0.

c $\int_b^a f(x) dx = F(a) - F(b)$
 $= -[F(b) - F(a)]$
 $= -\int_a^b f(x) dx$

b The antiderivative of c is cx .
 $\therefore \int_a^b c dx = F(b) - F(a)$
 $= cb - ca$
 $= c(b-a)$

d If $\frac{d}{dx} F(x) = f(x)$ then
 $\frac{d}{dx} c F(x) = c f(x)$
 $\therefore \int_a^b c f(x) dx = c F(b) - c F(a)$
 $= c[F(b) - F(a)]$
 $= c \int_a^b f(x) dx$

e $\int_a^b (f(x) + g(x)) dx = [F(b) + G(b)] - [F(a) + G(a)]$
 $= [F(b) - F(a)] + [G(b) - G(a)]$
 $= \int_a^b f(x) dx + \int_a^b g(x) dx$

2 a $f(x) = x^3$ has antiderivative $F(x) = \frac{x^4}{4}$
 \therefore area = $\int_0^1 x^3 dx$

$$\begin{aligned} &= F(1) - F(0) \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \text{ units}^2 \end{aligned}$$

b $f(x) = x^3$ has antiderivative $F(x) = \frac{x^4}{4}$
 \therefore area = $\int_1^2 x^3 dx$

$$\begin{aligned} &= F(2) - F(1) \\ &= \frac{16}{4} - \frac{1}{4} \\ &= 3\frac{3}{4} \text{ units}^2 \end{aligned}$$

c $f(x) = x^2 + 3x + 2$ has antiderivative

$$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} + 2x$$

\therefore area = $\int_1^3 (x^2 + 3x + 2) dx$
 $= F(3) - F(1)$
 $= (\frac{27}{3} + \frac{27}{2} + 6) - (\frac{1}{3} + \frac{3}{2} + 2)$
 $= 24\frac{2}{3} \text{ units}^2$

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$$

\therefore area = $\int_0^2 \sqrt{x} dx$
 $= F(2) - F(0)$
 $= \frac{2}{3} \times 2\sqrt{2} - 0$
 $= \frac{4\sqrt{2}}{3} \text{ units}^2$

Teacher

e $f(x) = e^x$ has antiderivative $F(x) = e^x$
 \therefore area $= \int_0^{1.5} e^x dx$
 $= F(1.5) - F(0)$
 $= e^{1.5} - e^0$
 $= e^{1.5} - 1$
 $\approx 3.48 \text{ units}^2$

g $f(x) = x^3 + 2x^2 + 7x + 4$ has antiderivative
 $F(x) = \frac{x^4}{4} + \frac{2x^3}{3} + \frac{7x^2}{2} + 4x$
 \therefore area $= \int_1^{1.25} (x^3 + 2x^2 + 7x + 4) dx$
 $= F(1.25) - F(1)$
 $= [12.38118 - 8.41667]$
 $\approx 3.96 \text{ units}^2$

3 Using technology:

a area $= \int_0^{1.5} e^{x^2} dx \approx 4.06 \text{ units}^2$

b area $= \int_2^4 (\ln x)^2 dx \approx 2.41 \text{ units}^2$

c area $= \int_1^2 \sqrt{9-x^2} dx \approx 2.58 \text{ units}^2$

4 a If $\frac{d}{dx} F(x) = f(x)$ then $\frac{d}{dx} (-F(x)) = -f(x)$
 $\therefore \int_a^b (-f(x)) dx = -F(b) - (-F(a))$
 $= -(F(b) - F(a))$
 $= -\int_a^b f(x) dx$

b Since $y = -f(x)$ is a reflection of $y = f(x)$ in the x -axis,
shaded area = area between the x -axis and $y = -f(x)$
from $x = a$ to $x = b$
 $= \int_a^b (-f(x)) dx$
 $= -\int_a^b f(x) dx$ {using a}

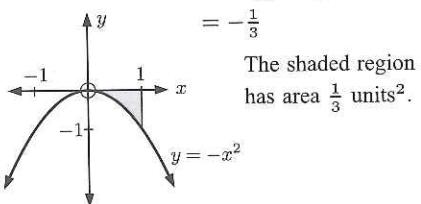
c i $\int_0^1 (-x^2) dx = -\int_0^1 x^2 dx$
Now $f(x) = x^2$ has antiderivative

$$F(x) = \frac{1}{3}x^3$$

$$\therefore \int_0^1 (-x^2) dx = -(F(1) - F(0))$$

$$= -\left(\frac{1}{3} - 0\right)$$

$$= -\frac{1}{3}$$



ii $\int_0^1 (x^2 - x) dx = -\int_0^1 (x - x^2) dx$

$\{x^2 - x \leq 0 \text{ for all } x \in [0, 1]\}$

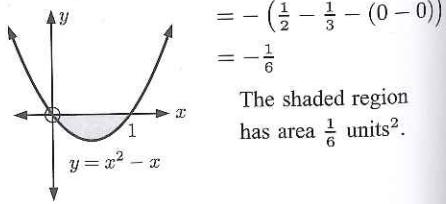
Now $f(x) = x - x^2$ has antiderivative

$$F(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

$$\therefore \int_0^1 (x^2 - x) dx = -(F(1) - F(0))$$

$$= -\left(\frac{1}{2} - \frac{1}{3} - (0 - 0)\right)$$

$$= -\frac{1}{6}$$



iii $\int_{-2}^0 3x dx = -\int_{-2}^0 -3x dx$

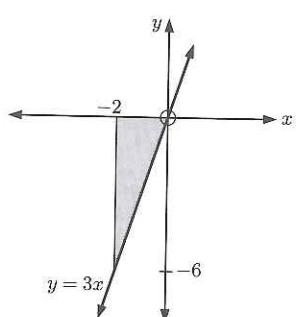
Now $f(x) = -3x$ has antiderivative

$$F(x) = -\frac{3}{2}x^2$$

$$\therefore \int_{-2}^0 3x dx = -(F(0) - F(-2))$$

$$= -(0 - (-6))$$

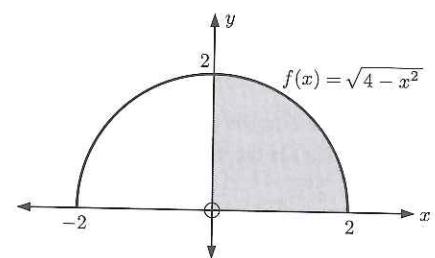
$$= -6$$



The shaded region has area 6 units².

d $\int_0^2 (-\sqrt{4-x^2}) dx = -\int_0^2 \sqrt{4-x^2} dx$

Now $f(x) = \sqrt{4-x^2}$ is the top half of a circle with radius 2 units and centre (0, 0).



$$\begin{aligned} \therefore \int_0^2 (-\sqrt{4-x^2}) dx &= -\int_0^2 \sqrt{4-x^2} dx \\ &= -(\text{shaded area}) \\ &= -\frac{1}{4} \times \pi \times 2^2 \\ &= -\pi \end{aligned}$$

EXERCISE 21C.1

1 If $y = x^7$ then $\frac{dy}{dx} = 7x^6$

$$\therefore \int 7x^6 dx = x^7 + c_1$$

$$\therefore 7 \int x^6 dx = x^7 + c_1$$

$$\therefore \int x^6 dx = \frac{1}{7}x^7 + c$$

3 If $y = e^{2x+1}$ then $\frac{dy}{dx} = 2e^{2x+1}$

$$\therefore \int 2e^{2x+1} dx = e^{2x+1} + c_1$$

$$\therefore 2 \int e^{2x+1} dx = e^{2x+1} + c_1$$

$$\therefore \int e^{2x+1} dx = \frac{1}{2}e^{2x+1} + c$$

2 If $y = x^3 + x^2$ then $\frac{dy}{dx} = 3x^2 + 2x$

$$\therefore \int (3x^2 + 2x) dx = x^3 + x^2 + c$$

4 If $y = (2x+1)^4$

then $\frac{dy}{dx} = 4(2x+1)^3 \times 2 = 8(2x+1)^3$

$$\therefore \int 8(2x+1)^3 dx = (2x+1)^4 + c_1$$

$$\therefore 8 \int (2x+1)^3 dx = (2x+1)^4 + c_1$$

$$\therefore \int (2x+1)^3 dx = \frac{1}{8}(2x+1)^4 + c$$

6 If $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

then $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$

$$\therefore \int -\frac{1}{2} \left(\frac{1}{x\sqrt{x}} \right) dx = \frac{1}{\sqrt{x}} + c_1$$

$$\therefore -\frac{1}{2} \int \frac{1}{x\sqrt{x}} dx = \frac{1}{\sqrt{x}} + c_1$$

$$\therefore \int \frac{1}{x\sqrt{x}} dx = -\frac{2}{\sqrt{x}} + c$$

Teacher

7 If $y = \cos 2x$
then $\frac{dy}{dx} = -2 \sin 2x$
 $\therefore \int -2 \sin 2x \, dx = \cos 2x + c_1$
 $\therefore -2 \int \sin 2x \, dx = \cos 2x + c_1$
 $\therefore \int \sin 2x \, dx = -\frac{1}{2} \cos 2x + c$

9 Suppose $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ is the antiderivative of $g(x)$.
 $\therefore \frac{d}{dx}(F(x) + G(x)) = f(x) + g(x)$
 $\therefore \int [f(x) + g(x)] \, dx = F(x) + G(x) + c$
 $= (F(x) + c_1) + (G(x) + c_2)$
 $= \int f(x) \, dx + \int g(x) \, dx$

11 $\frac{d}{dx}(\ln(5 - 3x + x^2)) = \frac{2x - 3}{5 - 3x + x^2}$
Now $5 - 3x + x^2 > 0$ for all x ,
as $a > 0$ and $\Delta = -11 < 0$.
 $\therefore \int \frac{2x - 3}{5 - 3x + x^2} \, dx = \ln(5 - 3x + x^2) + c_1$
 $\therefore \int \frac{4x - 6}{5 - 3x + x^2} \, dx = 2 \ln(5 - 3x + x^2) + c$

EXERCISE 21C.2

1 a $\int (x^4 - x^2 - x + 2) \, dx$
= $\frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c$
b $\int (\sqrt{x} + e^x) \, dx$
= $\int (x^{\frac{1}{2}} + e^x) \, dx$
= $\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + e^x + c$
c $\int \left(3e^x - \frac{1}{x}\right) \, dx$
= $3e^x - \ln x + c, x > 0$

d $\int \left(x\sqrt{x} - \frac{2}{x}\right) \, dx$
= $\int \left(x^{\frac{3}{2}} - \frac{2}{x}\right) \, dx$
= $\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2 \ln x + c, x > 0$
= $\frac{2}{5}x^{\frac{5}{2}} - 2 \ln x + c, x > 0$
e $\int \left(\frac{1}{x\sqrt{x}} + \frac{4}{x}\right) \, dx$
= $\int \left(x^{-\frac{3}{2}} + \frac{4}{x}\right) \, dx$
= $\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 4 \ln x + c, x > 0$
= $-\frac{2}{\sqrt{x}} + 4 \ln x + c, x > 0$
f $\int (\frac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}) \, dx$
= $\frac{1}{2} \cdot \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c$
= $\frac{1}{8}x^4 - \frac{1}{5}x^5 + \frac{3}{4}x^{\frac{4}{3}} + c$

g $\int \left(x^2 + \frac{3}{x}\right) \, dx$
= $\frac{1}{3}x^3 + 3 \ln x + c, x > 0$
h $\int \left(\frac{1}{2x} + x^2 - e^x\right) \, dx$
= $\frac{1}{2} \ln x + \frac{1}{3}x^3 - e^x + c, x > 0$

8 If $y = \sin(1 - 5x)$
then $\frac{dy}{dx} = -5 \cos(1 - 5x)$
 $\therefore \int -5 \cos(1 - 5x) \, dx = \sin(1 - 5x) + c_1$
 $\therefore -5 \int \cos(1 - 5x) \, dx = \sin(1 - 5x) + c_1$
 $\therefore \int \cos(1 - 5x) \, dx = -\frac{1}{5} \sin(1 - 5x) + c$

10 $y = \sqrt{1 - 4x} = (1 - 4x)^{\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = \frac{1}{2}(1 - 4x)^{-\frac{1}{2}}(-4)$
= $\frac{-2}{\sqrt{1 - 4x}}$
 $\therefore \int \frac{-2}{\sqrt{1 - 4x}} \, dx = \sqrt{1 - 4x} + c_1$
 $\therefore -2 \int \frac{1}{\sqrt{1 - 4x}} \, dx = \sqrt{1 - 4x} + c_1$
 $\therefore \int \frac{1}{\sqrt{1 - 4x}} \, dx = -\frac{1}{2}\sqrt{1 - 4x} + c$

i $\int \left(5e^x + \frac{1}{3}x^3 - \frac{4}{x}\right) \, dx$
= $5e^x + \frac{1}{3} \cdot \frac{x^4}{4} - 4 \ln x + c, x > 0$
= $5e^x + \frac{1}{12}x^4 - 4 \ln x + c, x > 0$

2 a $\int (3 \sin x - 2) \, dx$
= $-3 \cos x - 2x + c$
b $\int (4x - 2 \cos x) \, dx$
= $2x^2 - 2 \sin x + c$
c $\int (\sin x - 2 \cos x + e^x) \, dx$
= $-\cos x - 2 \sin x + e^x + c$
d $\int (x^2 \sqrt{x} - 10 \sin x) \, dx$
= $\int (x^{\frac{5}{2}} - 10 \sin x) \, dx$
= $\frac{2}{7}x^{\frac{7}{2}} + 10 \cos x + c$
= $\frac{2}{7}x^3 \sqrt{x} + 10 \cos x + c$
e $\int \left(\frac{x(x-1)}{3} + \cos x\right) \, dx$
= $\int \left(\frac{x^2}{3} - \frac{x}{3} + \cos x\right) \, dx$
= $\frac{x^3}{9} - \frac{x^2}{6} + \sin x + c$
f $\int (-\sin x + 2\sqrt{x}) \, dx$
= $\int (-\sin x + 2x^{\frac{1}{2}}) \, dx$
= $\cos x + \frac{4}{3}x^{\frac{3}{2}} + c$
= $\cos x + \frac{4}{3}x\sqrt{x} + c$

3 a $\int (x^2 + 3x - 2) \, dx$
= $\frac{x^3}{3} + \frac{3x^2}{2} - 2x + c$
b $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) \, dx$
= $\int \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) \, dx$
= $\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
= $\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$
c $\int \left(2e^x - \frac{1}{x^2}\right) \, dx$
= $\int (2e^x - x^{-2}) \, dx$
= $2e^x - \frac{x^{-1}}{-1} + c$
= $2e^x + \frac{1}{x} + c$

d $\int \left(\frac{1-4x}{x\sqrt{x}}\right) \, dx$
= $\int \left(\frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}}\right) \, dx$
= $\int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}}) \, dx$
= $\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$
= $-2x^{-\frac{1}{2}} - 8x^{\frac{1}{2}} + c$
e $\int (2x+1)^2 \, dx$
= $\int (4x^2 + 4x + 1) \, dx$
= $\frac{4x^3}{3} + \frac{4x^2}{2} + x + c$
= $\frac{4}{3}x^3 + 2x^2 + x + c$
f $\int \frac{x^2+x-3}{x} \, dx$
= $\int (x+1 - \frac{3}{x}) \, dx$
= $\frac{x^2}{2} + x - 3 \ln x + c, x > 0$

g $\int \frac{2x-1}{\sqrt{x}} \, dx$
= $\int \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) \, dx$
= $\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
= $\frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$
h $\int \frac{x^2-4x+10}{x^2\sqrt{x}} \, dx$
= $\int \left(\frac{x^2}{x^2\sqrt{x}} - \frac{4x}{x^2\sqrt{x}} + \frac{10}{x^2\sqrt{x}}\right) \, dx$
= $\int \left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}} + 10x^{-\frac{5}{2}}\right) \, dx$
= $\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{10x^{-\frac{3}{2}}}{-\frac{3}{2}} + c$
= $2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}} - \frac{20}{3}x^{-\frac{3}{2}} + c$

Teacher

4 **a** $\int (\sqrt{x} + \frac{1}{2} \cos x) dx$

$$\begin{aligned} &= \int (x^{\frac{1}{2}} + \frac{1}{2} \cos x) dx \\ &= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2} \sin x + c \end{aligned}$$

b $\int (2e^t - 4 \sin t) dt$

$$\begin{aligned} &= 2e^t + 4 \cos t + c \\ &= 2e^t + 4 \cos t + c \end{aligned}$$

c $\int \left(3 \cos t - \frac{1}{t}\right) dt$

$$\begin{aligned} &= 3 \sin t - \ln t + c, \quad t > 0 \\ &= 3 \sin t - \ln t + c, \quad t > 0 \end{aligned}$$

5 **a** $\frac{dy}{dx} = 6$

$$\begin{aligned} \therefore y &= \int 6 dx \\ \therefore y &= 6x + c \end{aligned}$$

b $\frac{dy}{dx} = 4x^2$

$$\begin{aligned} \therefore y &= \int 4x^2 dx \\ \therefore y &= \frac{4}{3}x^3 + c \end{aligned}$$

c $\frac{dy}{dx} = 5\sqrt{x} - x^2 = 5x^{\frac{1}{2}} - x^2$

$$\begin{aligned} \therefore y &= \int (5x^{\frac{1}{2}} - x^2) dx \\ \therefore y &= \frac{10}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 + c \\ \therefore y &= \frac{10}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 + c \end{aligned}$$

d $\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$

$$\begin{aligned} \therefore y &= \int x^{-2} dx \\ \therefore y &= \frac{x^{-1}}{-1} + c \\ \therefore y &= -\frac{1}{x} + c \end{aligned}$$

e $\frac{dy}{dx} = 2e^x - 5$

$$\begin{aligned} \therefore y &= \int (2e^x - 5) dx \\ \therefore y &= 2e^x - 5x + c \end{aligned}$$

f $\frac{dy}{dx} = 4x^3 + 3x^2$

$$\begin{aligned} \therefore y &= \int (4x^3 + 3x^2) dx \\ \therefore y &= \frac{4x^4}{4} + \frac{3x^3}{3} + c \\ \therefore y &= x^4 + x^3 + c \end{aligned}$$

6 **a** $\frac{dy}{dx} = (1 - 2x)^2$

$$\begin{aligned} \therefore y &= \int (1 - 2x)^2 dx \\ &= \int (1 - 4x + 4x^2) dx \\ &= x - \frac{4x^2}{2} + \frac{4x^3}{3} + c \\ &= x - 2x^2 + \frac{4}{3}x^3 + c \end{aligned}$$

b $\frac{dy}{dx} = \sqrt{x} - \frac{2}{\sqrt{x}}$

$$\begin{aligned} &= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\ \therefore y &= \int (x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c \end{aligned}$$

c $\frac{dy}{dx} = \frac{x^2 + 2x - 5}{x^2}$

$$\begin{aligned} &= 1 + 2x^{-1} - 5x^{-2} \\ \therefore y &= \int (1 + 2x^{-1} - 5x^{-2}) dx \\ &= x + 2 \ln x - \frac{5x^{-1}}{-1} + c, \quad x > 0 \\ &= x + 2 \ln x + \frac{5}{x} + c, \quad x > 0 \end{aligned}$$

7 **a** $f'(x) = x^3 - 5\sqrt{x} + 3$

$$\begin{aligned} &= x^3 - 5x^{\frac{1}{2}} + 3 \\ \therefore f(x) &= \int (x^3 - 5x^{\frac{1}{2}} + 3) dx \\ &= \frac{1}{4}x^4 - \frac{10}{3}x^{\frac{3}{2}} + 3x + c \end{aligned}$$

c $f'(x) = 3e^x - \frac{4}{x}$

$$\begin{aligned} \therefore f(x) &= \int \left(3e^x - \frac{4}{x}\right) dx \\ &= 3e^x - 4 \ln x + c, \quad x > 0 \end{aligned}$$

b $f'(x) = 2\sqrt{x}(1 - 3x)$

$$\begin{aligned} &= 2x^{\frac{1}{2}} - 6x^{\frac{3}{2}} \\ \therefore f(x) &= \int (2x^{\frac{1}{2}} - 6x^{\frac{3}{2}}) dx \\ &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{6x^{\frac{5}{2}}}{\frac{5}{2}} + c \\ &= \frac{4}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{2}} + c \end{aligned}$$

8 **a** $f'(x) = 2x - 1$

$$\begin{aligned} \therefore f(x) &= \int (2x - 1) dx \\ &= \frac{2x^2}{2} - x + c \\ &= x^2 - x + c \\ \text{But } f(0) &= 3, \text{ so } 0 - 0 + c = 3 \\ \therefore c &= 3 \\ \therefore f(x) &= x^2 - x + 3 \\ \text{c} \quad f'(x) &= e^x + \frac{1}{\sqrt{x}} = e^x + x^{-\frac{1}{2}} \\ \therefore f(x) &= \int (e^x + x^{-\frac{1}{2}}) dx \\ &= e^x + 2x^{\frac{1}{2}} + c \\ \text{But } f(1) &= 1, \text{ so } e^1 + 2 + c = 1 \\ \therefore c &= -1 - e \\ \therefore f(x) &= e^x + 2\sqrt{x} - 1 - e \\ \text{d} \quad f'(x) &= x - \frac{2}{\sqrt{x}} = x - 2x^{-\frac{1}{2}} \\ \therefore f(x) &= \int (x - 2x^{-\frac{1}{2}}) dx \\ &= \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{1}{2}x^2 - 4\sqrt{x} + c \\ \text{But } f(1) &= 2, \text{ so } \frac{1}{2} - 4 + c = 2 \\ \therefore c &= \frac{11}{2} \\ \therefore f(x) &= \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2} \end{aligned}$$

9 **a** $f'(x) = x^2 - 4 \cos x$

$$\begin{aligned} \therefore f(x) &= \int (x^2 - 4 \cos x) dx \\ &= \frac{x^3}{3} - 4 \sin x + c \\ \text{But } f(0) &= 3 \\ \therefore 0 - 4 \sin(0) + c &= 3 \\ \therefore c &= 3 \\ \therefore f(x) &= \frac{x^3}{3} - 4 \sin x + 3 \end{aligned}$$

b $f'(x) = 2 \cos x - 3 \sin x$

$$\begin{aligned} \therefore f(x) &= \int (2 \cos x - 3 \sin x) dx \\ &= 2 \sin x + 3 \cos x + c \\ \text{But } f(\frac{\pi}{4}) &= \frac{1}{\sqrt{2}} \\ \therefore 2 \sin \frac{\pi}{4} + 3 \cos \frac{\pi}{4} + c &= \frac{1}{\sqrt{2}} \\ \therefore 2(\frac{1}{\sqrt{2}}) + 3(\frac{1}{\sqrt{2}}) + c &= \frac{1}{\sqrt{2}} \\ \therefore c &= -\frac{4}{\sqrt{2}} \\ \therefore c &= -2\sqrt{2} \\ \therefore f(x) &= 2 \sin x + 3 \cos x - 2\sqrt{2} \end{aligned}$$

10 **a** Given: $f''(x) = 2x + 1$, $f'(1) = 3$, $f(2) = 7$

$$\begin{aligned} \therefore f'(x) &= \int (2x + 1) dx \\ &= \frac{2x^2}{2} + x + c \\ &= x^2 + x + c \\ \text{But } f'(1) &= 3 \text{ so } 1 + 1 + c = 3 \\ \therefore c &= 1 \\ \therefore f'(x) &= x^2 + x + 1 \end{aligned}$$

Then $f(x) = \int (x^2 + x + 1) dx$

$$\begin{aligned} &= \frac{x^3}{3} + \frac{x^2}{2} + x + k \\ \text{But } f(2) &= 7 \text{ so } \frac{8}{3} + 2 + 2 + k = 7 \\ \therefore k &= 7 - 4 - \frac{8}{3} \\ \therefore k &= \frac{1}{3} \\ \therefore f(x) &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3} \end{aligned}$$

b Given: $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}}$, $f'(1) = 12$, $f(0) = 5$

Now $f''(x) = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$$\therefore f'(x) = \frac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 10x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + c$$

But $f'(1) = 12$ so $10 + 6 + c = 12$

$$\therefore c = -4$$

$\therefore f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$

$$\therefore f(x) = \int (10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4) dx$$

$$= \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + k$$

$$= 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + k$$

But $f(0) = 5$ so $k = 5$

$$\therefore f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$$

c Given: $f''(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$ and $f(0) = 3$

Now $f'(x) = \int \cos x dx = \sin x + c$

But $f'(\frac{\pi}{2}) = 0$ so $\sin(\frac{\pi}{2}) + c = 0$

$$\therefore c = -1$$

$\therefore f'(x) = \sin x - 1$

So, $f(x) = \int (\sin x - 1) dx$

$$= -\cos x - x + k$$

But $f(0) = 3$ so $-\cos 0 - 0 + k = 3$

$$\therefore -1 + k = 3$$

$$\therefore k = 4$$

So, $f(x) = -\cos x - x + 4$

d Given: $f''(x) = 2x$ and that $(1, 0)$ and $(0, 5)$ lie on the curve

Now $f'(x) = \int 2x dx = \frac{2x^2}{2} + c = x^2 + c$

$$\therefore f(x) = \int (x^2 + c) dx = \frac{x^3}{3} + cx + k$$

But $f(0) = 5$ so $0 + 0 + k = 5$ and so $k = 5$

and $f(1) = 0$ so $\frac{1}{3} + c + 5 = 0$ and so $c = -5\frac{1}{3}$

$$\therefore f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$$

EXERCISE 21D

1 a $\int (2x+5)^3 dx$

$$= \frac{1}{2} \times \frac{(2x+5)^4}{4} + c$$

$$= \frac{1}{8}(2x+5)^4 + c$$

b $\int \frac{1}{(3-2x)^2} dx$

$$= \int (3-2x)^{-2} dx$$

$$= \frac{1}{-2} \times \frac{(3-2x)^{-1}}{-1} + c$$

$$= \frac{1}{2(3-2x)} + c$$

c $\int \frac{4}{(2x-1)^4} dx$

$$= \int 4(2x-1)^{-4} dx$$

$$= 4(\frac{1}{2}) \times \frac{(2x-1)^{-3}}{-3} + c$$

$$= \frac{-2}{3(2x-1)^3} + c$$

d $\int (4x-3)^7 dx$

$$= \frac{1}{4} \times \frac{(4x-3)^8}{8} + c$$

$$= \frac{1}{32}(4x-3)^8 + c$$

e $\int \sqrt{3x-4} dx$

$$= \int (3x-4)^{\frac{1}{2}} dx$$

$$= \frac{1}{3} \times \frac{(3x-4)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{9}(3x-4)^{\frac{3}{2}} + c$$

f $\int \frac{10}{\sqrt{1-5x}} dx$

$$= \int 10(1-5x)^{-\frac{1}{2}} dx$$

$$= 10(\frac{1}{-5}) \times \frac{(1-5x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -4\sqrt{1-5x} + c$$

g $\int 3(1-x)^4 dx$

$$= 3 \int (1-x)^4 dx$$

$$= 3(\frac{1}{-1}) \times \frac{(1-x)^5}{5} + c$$

$$= -\frac{3}{5}(1-x)^5 + c$$

h $\int \frac{4}{\sqrt{3-4x}} dx$

$$= \int 4(3-4x)^{-\frac{1}{2}} dx$$

$$= 4(\frac{1}{-4}) \times \frac{(3-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -2\sqrt{3-4x} + c$$

2 a $\int \sin(3x) dx$

$$= -\frac{1}{3} \cos(3x) + c$$

b $\int 2 \cos(-4x) + 1 dx$

$$= 2 \times (\frac{1}{-4}) \sin(-4x) + x + c$$

$$= -\frac{1}{2} \sin(-4x) + x + c$$

c $\int 3 \cos(\frac{x}{2}) dx$

$$= 6 \sin(\frac{x}{2}) + c$$

d $\int (3 \sin(2x) - e^{-x}) dx$

$$= -\frac{3}{2} \cos(2x) + e^{-x} + c$$

e $\int 2 \sin(2x + \frac{\pi}{6}) dx$

$$= -\frac{2}{2} \cos(2x + \frac{\pi}{6}) + c$$

$$= -\cos(2x + \frac{\pi}{6}) + c$$

f $\int -3 \cos(\frac{\pi}{4} - x) dx$

$$= -3 \times (-1) \sin(\frac{\pi}{4} - x) + c$$

$$= 3 \sin(\frac{\pi}{4} - x) + c$$

g $\int \cos(2x) + \sin(2x) dx$

$$= \frac{1}{2} \sin(2x) - \frac{1}{2} \cos(2x) + c$$

h $\int 2 \sin(3x) + 5 \cos(4x) dx$

$$= -\frac{2}{3} \cos(3x) + \frac{5}{4} \sin(4x) + c$$

i $\int (\frac{1}{2} \cos(8x) - 3 \sin x) dx$

$$= \frac{1}{2}(\frac{1}{8}) \sin(8x) + 3 \cos x + c$$

$$= \frac{1}{16} \sin(8x) + 3 \cos x + c$$

3 a $\frac{dy}{dx} = \sqrt{2x-7} = (2x-7)^{\frac{1}{2}}$

$$\therefore y = \frac{1}{2} \times \frac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}(2x-7)^{\frac{3}{2}} + c$$

But $y = 11$ when $x = 8$

$$\therefore \frac{1}{3}(16-7)^{\frac{3}{2}} + c = 11$$

$$\therefore \frac{1}{3}(27) + c = 11$$

$$\therefore 9 + c = 11 \text{ and so } c = 2$$

$$\therefore y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$$

b $f(x)$ has gradient function $f'(x) = \frac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$

$$\therefore f(x) = 4(\frac{1}{-1}) \times \frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -8\sqrt{1-x} + c$$

But $y = -11$ when $x = -3$

$$\therefore -8\sqrt{1-(-3)} + c = -11$$

$$\therefore -8\sqrt{4} + c = -11$$

$$\therefore -16 + c = -11 \text{ and so } c = 5$$

$$\therefore f(x) = 5 - 8\sqrt{1-x}$$

Now $f(-8) = 5 - 8\sqrt{1-(-8)} = 5 - 8(3) = -19$, so the point is $(-8, -19)$.

4 a $\int \cos^2 x dx$

$$= \int (\frac{1}{2} + \frac{1}{2} \cos(2x)) dx$$

$$= \frac{1}{2}x + \frac{1}{4} \sin(2x) + c$$

b $\int \sin^2 x dx$

$$= \int (\frac{1}{2} - \frac{1}{2} \cos(2x)) dx$$

$$= \frac{1}{2}x - \frac{1}{4} \sin(2x) + c$$

Teacher

c $\int (1 + \cos^2(2x)) dx$
 $= \int (1 + \frac{1}{2} + \frac{1}{2} \cos(4x)) dx$
 $= \int (\frac{3}{2} + \frac{1}{2} \cos(4x)) dx$
 $= \frac{3}{2}x + \frac{1}{8} \sin(4x) + c$

e $\int \frac{1}{2} \cos^2(4x) dx$
 $= \int \frac{1}{2}(\frac{1}{2} + \frac{1}{2} \cos(8x)) dx$
 $= \int (\frac{1}{4} + \frac{1}{4} \cos(8x)) dx$
 $= \frac{1}{4}x + \frac{1}{32} \sin(8x) + c$

5 a $\int 3(2x-1)^2 dx$
 $= 3 \int (2x-1)^2 dx$
 $= 3(\frac{1}{2}) \frac{(2x-1)^3}{3} + c$
 $= \frac{1}{2}(2x-1)^3 + c$

d $\int (1-x^2)^2 dx$
 $= \int (1-2x^2+x^4) dx$
 $= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c$

6 a $\int (2e^x + 5e^{2x}) dx$
 $= 2e^x + 5(\frac{1}{2})e^{2x} + c$
 $= 2e^x + \frac{5}{2}e^{2x} + c$

d $\int \frac{1}{2x-1} dx$
 $= \frac{1}{2} \ln(2x-1) + c, \quad 2x-1 > 0$
 $= \frac{1}{2} \ln(2x-1) + c, \quad x > \frac{1}{2}$

f $\int \left(e^{-x} - \frac{4}{2x+1}\right) dx$
 $= \frac{1}{-1}e^{-x} - 4(\frac{1}{2}) \ln(2x+1) + c, \quad 2x+1 > 0$
 $= -e^{-x} - 2 \ln(2x+1) + c, \quad x > -\frac{1}{2}$

h $\int (e^{-x} + 2)^2 dx$
 $= \int (e^{-2x} + 4e^{-x} + 4) dx$
 $= \frac{1}{-2}e^{-2x} + 4(\frac{1}{-1})e^{-x} + 4x + c$
 $= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

d $\int (3 - \sin^2(3x)) dx$
 $= \int (3 - (\frac{1}{2} - \frac{1}{2} \cos(6x))) dx$
 $= \int (\frac{5}{2} + \frac{1}{2} \cos(6x)) dx$
 $= \frac{5}{2}x + \frac{1}{12} \sin(6x) + c$

f $\int (1 + \cos x)^2 dx$
 $= \int (1 + 2 \cos x + \cos^2 x) dx$
 $= \int (1 + 2 \cos x + \frac{1}{2} + \frac{1}{2} \cos(2x)) dx$
 $= \int (\frac{3}{2} + 2 \cos x + \frac{1}{2} \cos(2x)) dx$
 $= \frac{3}{2}x + 2 \sin x + \frac{1}{4} \sin(2x) + c$

b $\int (x^2 - x)^2 dx$
 $= \int (x^4 - 2x^3 + x^2) dx$
 $= \frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + c$
 $= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

e $\int 4\sqrt{5-x} dx$
 $= 4 \int (5-x)^{\frac{1}{2}} dx$
 $= 4(\frac{1}{\frac{3}{2}}) \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c$
 $= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c$

f $\int (x^2 + 1)^3 dx$
 $= \int (x^6 + 3x^4 + 3x^2 + 1) dx$
 $= \frac{x^7}{7} + \frac{3x^5}{5} + \frac{3x^3}{3} + x + c$
 $= \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

c $\int (1-3x)^3 dx$
 $= (\frac{1}{-3}) \frac{(1-3x)^4}{4} + c$
 $= -\frac{1}{12}(1-3x)^4 + c$

b $\int (3e^{5x-2}) dx$
 $= 3(\frac{1}{5})e^{5x-2} + c$
 $= \frac{3}{5}e^{5x-2} + c$

e $\int \frac{5}{1-3x} dx$
 $= 5 \int \frac{1}{1-3x} dx$
 $= 5(\frac{1}{-3}) \ln(1-3x) + c, \quad 1-3x > 0$
 $= -\frac{5}{3} \ln(1-3x) + c, \quad x < \frac{1}{3}$

g $\int (e^x + e^{-x})^2 dx$
 $= \int (e^{2x} + 2 + e^{-2x}) dx$
 $= \frac{1}{2}e^{2x} + 2x + (\frac{1}{-2})e^{-2x} + c$
 $= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

i $\int \left(x - \frac{5}{1-x}\right) dx$
 $= \frac{x^2}{2} - 5(\frac{1}{-1}) \ln(1-x) + c, \quad 1-x > 0$
 $= \frac{1}{2}x^2 + 5 \ln(1-x) + c, \quad x < 1$

7 a $\frac{dy}{dx} = (1-e^x)^2$
 $= 1-2e^x + e^{2x}$
 $\therefore y = x - 2e^x + \frac{1}{2}e^{2x} + c$
 $\therefore y = x - x^2 + 3 \ln(x+2) + c, \quad x+2 > 0$
 $\therefore y = x - x^2 + 3 \ln(x+2) + c, \quad x > -2$

c $\frac{dy}{dx} = e^{-2x} + \frac{4}{2x-1}$
 $\therefore y = \frac{1}{-2}e^{-2x} + 4(\frac{1}{2}) \ln(2x-1) + c, \quad 2x-1 > 0$
 $\therefore -\frac{1}{2}e^{-2x} + 2 \ln(2x-1) + c, \quad x > \frac{1}{2}$

8 Differentiating Tracy's answer gives

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{4} \ln(4x) + c \right) &= \frac{1}{4} \left(\frac{1}{4x} \right) \times 4 + 0, \quad x > 0 \\ &= \frac{1}{4x}, \quad x > 0 \\ \frac{d}{dx} \left(\frac{1}{4} \ln(x) + c \right) &= \frac{1}{4} \left(\frac{1}{x} \right) + 0, \quad x > 0 \\ &= \frac{1}{4x}, \quad x > 0 \end{aligned}$$

Both answers give the correct derivative and both are correct. This result occurs because $\ln(4x) = \ln 4 + \ln x$. Their answers differ by a constant which is accounted for by c .

9 Given: $f'(x) = p \sin(\frac{1}{2}x)$, $f(0) = 1$ and $f(2\pi) = 0$
 $\therefore f(x) = -2p \cos(\frac{1}{2}x) + c$

But $f(0) = 1$, so $-2p \cos(0) + c = 1$
 $\therefore -2p + c = 1$

$\therefore c = 1 + 2p \quad \dots (1)$

Also, $f(2\pi) = 0$, so $-2p \cos(\pi) + c = 0$

$\therefore 2p + c = 0$

$\therefore 2p + 1 + 2p = 0 \quad \{ \text{using (1)} \}$

$$\begin{aligned} \therefore p &= -\frac{1}{4} \\ \therefore c &= \frac{1}{2} \quad \{ \text{from (1)} \} \\ \therefore f(x) &= \frac{1}{2} \cos(\frac{1}{2}x) + \frac{1}{2} \end{aligned}$$

10 $g''(x) = -\sin 2x$

Integrating both sides with respect to x , we get $g'(x) = \frac{1}{2} \cos 2x + c$, c some constant.

So, $g'(\pi) = \frac{1}{2} \cos(2\pi) + c$ and $g'(-\pi) = \frac{1}{2} \cos(-2\pi) + c$
 $= \frac{1}{2} + c$
 $= \frac{1}{2} + c$
 $= g'(\pi)$

 \therefore the gradients of the tangents to $y = g(x)$ at $x = \pi$ and $x = -\pi$ are equal.

11 a $f'(x) = 2e^{-2x}$
 $\therefore f(x) = 2(\frac{1}{-2})e^{-2x} + c$
 $= -e^{-2x} + c$

But $f(0) = 3$ so $-e^0 + c = 3$
 $\therefore c = 4$
 $\therefore f(x) = -e^{-2x} + 4$

b $f'(x) = 2x - \frac{2}{1-x}$
 $\therefore f(x) = \frac{2x^2}{2} - \frac{2}{-1} \ln(1-x) + c, \quad 1-x > 0$
 $= x^2 + 2 \ln(1-x) + c, \quad x < 1$

But $f(-1) = 3$ so $1+2 \ln(2)+c=3$
 $\therefore c=2-2 \ln 2$
 $\therefore f(x)=x^2+2 \ln(1-x)+2-2 \ln 2, \quad x<1$

c $f'(x) = \sqrt{x} + \frac{1}{2}e^{-4x}$
 $= x^{\frac{1}{2}} + \frac{1}{2}e^{-4x}$
 $\therefore f(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}(\frac{1}{-4})e^{-4x} + c$
 $= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + c$

But $f(1) = 0$
 $\therefore \frac{2}{3} - \frac{1}{8}e^{-4} + c = 0$
 $\therefore c = \frac{1}{8}e^{-4} - \frac{2}{3}$
 $\therefore f(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

12 $(\sin x + \cos x)^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x$
 $= 1 + \sin 2x$
 $\therefore \int (\sin x + \cos x)^2 dx = \int (1 + \sin 2x) dx$
 $= x - \frac{1}{2} \cos 2x + c$

13 $(\cos x + 1)^2 = \cos^2 x + 2 \cos x + 1$
 $= (\frac{1}{2} + \frac{1}{2} \cos 2x) + 2 \cos x + 1$
 $= \frac{1}{2} \cos 2x + 2 \cos x + \frac{3}{2}$
 $\therefore \int (\cos x + 1)^2 dx = \int (\frac{1}{2} \cos 2x + 2 \cos x + \frac{3}{2}) dx$
 $= \frac{1}{4} \sin 2x + 2 \sin x + \frac{3}{2}x + c$

EXERCISE 21E.1

1 a $\int_0^1 x^3 dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

b $\int_0^2 (x^2 - x) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^2 = (\frac{8}{3} - 2) - (0 - 0) = \frac{2}{3}$

c $\int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = e - 1 \approx 1.72$

d $\int_0^{\frac{\pi}{6}} \cos x dx = [\sin x]_0^{\frac{\pi}{6}} = \sin \frac{\pi}{6} - \sin 0 = \frac{1}{2}$

e $\int_1^4 \left(x - \frac{3}{\sqrt{x}} \right) dx = \int_1^4 (x - 3x^{-\frac{1}{2}}) dx = \left[\frac{x^2}{2} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 = \left[\frac{x^2}{2} - 6\sqrt{x} \right]_1^4 = \left[\frac{16}{2} - 12 \right] - \left(\frac{1}{2} - 6 \right) = 1\frac{1}{2}$

f $\int_4^9 \frac{x-3}{\sqrt{x}} dx = \int_4^9 (x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_4^9 = \left[\frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} \right]_4^9 = \left[\frac{2}{3}(27) - 6(3) \right] - \left[\frac{2}{3}(8) - 6(2) \right] = (18 - 18) - (\frac{16}{3} - 12) = 6\frac{2}{3}$

g $\int_1^3 \frac{1}{x} dx = [\ln(x)]_1^3 = \ln 3 - \ln 1 = \ln 3 - 0 = \ln 3 \approx 1.10$

h $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x dx = [-\cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -\cos \frac{\pi}{2} + \cos \frac{\pi}{3} = \frac{1}{2}$

i $\int_1^2 (e^{-x} + 1)^2 dx = \int_1^2 (e^{-2x} + 2e^{-x} + 1) dx = \left[(\frac{1}{-2})e^{-2x} + 2(\frac{1}{-1})e^{-x} + x \right]_1^2 = \left[-\frac{e^{-2x}}{2} - 2e^{-x} + x \right]_1^2 = \left(-\frac{e^{-4}}{2} - 2e^{-2} + 2 \right) - \left(-\frac{e^{-2}}{2} - 2e^{-1} + 1 \right) \approx 1.52$

j $\int_2^6 \frac{1}{\sqrt{2x-3}} dx = \int_2^6 (2x-3)^{-\frac{1}{2}} dx = \left[\frac{1}{2} \frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^6 = \left[\sqrt{2x-3} \right]_2^6 = \sqrt{9} - \sqrt{1} = 2$

k $\int_0^1 e^{1-x} dx = \left[(\frac{1}{-1})e^{1-x} \right]_0^1 = \left(\frac{e^0}{-1} \right) - \left(\frac{e^1}{-1} \right) = -1 + e \approx 1.72$

l $\int_0^{\frac{\pi}{6}} \sin(3x) dx = \left[-\frac{1}{3} \cos(3x) \right]_0^{\frac{\pi}{6}} = -\frac{1}{3}[\cos \frac{\pi}{2} - \cos 0] = -\frac{1}{3}[0 - 1] = \frac{1}{3}$

m $\int_0^{\frac{\pi}{4}} \cos^2 x dx = \int_0^{\frac{\pi}{4}} (\frac{1}{2} + \frac{1}{2} \cos(2x)) dx = \left[\frac{x}{2} + \frac{1}{4} \sin(2x) \right]_0^{\frac{\pi}{4}} = \left[\frac{\pi}{8} + \frac{1}{4} \sin \frac{\pi}{2} \right] - 0 = \frac{\pi}{8} + \frac{1}{4}$

n $\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} (\frac{1}{2} - \frac{1}{2} \cos(2x)) dx = \left[\frac{x}{2} - \frac{1}{4} \sin(2x) \right]_0^{\frac{\pi}{2}} = \left[\frac{\pi}{4} - \frac{1}{4} \sin \pi \right] - 0 = \frac{\pi}{4}$

2 Using technology:

a $\int_1^3 \ln x dx \approx 1.30$

b $\int_{-1}^1 e^{-x^2} dx \approx 1.49$

c $\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \sin(\sqrt{x}) dx \approx -0.189$

EXERCISE 21E.2

1 a $\int_1^4 \sqrt{x} dx = \int_1^4 x^{\frac{1}{2}} dx = \left[\frac{2}{3}x^{\frac{3}{2}} \right]_1^4 = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{14}{3}$

b $\int_0^1 x^7 dx = \left[\frac{1}{8}x^8 \right]_0^1 = \frac{1}{8} - 0 = \frac{1}{8}$

$\int_0^1 (-x^7) dx = \left[-\frac{1}{8}x^8 \right]_0^1 = -\frac{1}{8} - 0 = -\frac{1}{8}$

Property: $\int_a^b [-f(x)] dx = - \int_a^b f(x) dx$

2 a $\int_0^1 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

b $\int_1^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_1^2 = \frac{1}{3}(8) - \frac{1}{3}(1) = \frac{7}{3}$

c $\int_0^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^2 = \frac{1}{3}(8) - 0 = \frac{8}{3}$

d $\int_0^1 3x^2 dx = \left[x^3 \right]_0^1 = 1 - 0 = 1$

Teacher

Properties: $\int_a^b f(x) dx + \int_a^c f(x) dx = \int_a^c f(x) dx$
 $\int_a^b c f(x) dx = c \int_a^b f(x) dx, \quad c \text{ a constant}$

3 a $\int_0^2 (x^3 - 4x) dx = [\frac{1}{4}x^4 - 2x^2]_0^2$
 $= [\frac{1}{4}(16) - 2(4)] - [0 - 0]$
 $= -4$

b $\int_2^3 (x^3 - 4x) dx = [\frac{1}{4}x^4 - 2x^2]_2^3$
 $= [\frac{1}{4}(81) - 2(9)] - [\frac{1}{4}(16) - 2(4)]$
 $= \frac{25}{4}$

c $\int_0^3 (x^3 - 4x) dx = [\frac{1}{4}x^4 - 2x^2]_0^3$
 $= [\frac{1}{4}(81) - 2(9)] - [0 - 0]$
 $= \frac{9}{4}$

4 a $\int_0^1 x^2 dx = [\frac{1}{3}x^3]_0^1$
 $= \frac{1}{3}(1) - 0$
 $= \frac{1}{3}$

b $\int_0^1 \sqrt{x} dx = \int_0^1 x^{\frac{1}{2}} dx$
 $= [\frac{2}{3}x^{\frac{3}{2}}]_0^1$
 $= \frac{2}{3}(1) - 0$
 $= \frac{2}{3}$

c $\int_0^1 (x^2 + \sqrt{x}) dx$
 $= \int_0^1 (x^2 + x^{\frac{1}{2}}) dx$
 $= [\frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}}]_0^1$
 $= [\frac{1}{3}(1) + \frac{2}{3}(1)] - [0 + 0]$
 $= 1$

Property: $\int_a^b f(x) dx + \int_a^b g(x) dx = \int_a^b [f(x) + g(x)] dx$

5 a $\int_0^3 f(x) dx = \text{area between } f(x) \text{ and the } x\text{-axis from } x = 0 \text{ to } x = 3$
 $= 2 + 3 + 1.5 = 6.5$

b $\int_3^7 f(x) dx = -(\text{area between } f(x) \text{ and the } x\text{-axis from } x = 3 \text{ to } x = 7)$
 $= -(\frac{3}{2} + 3 + \frac{5}{2} + 2) = -9$

c $\int_2^4 f(x) dx = (\text{area between } f(x) \text{ and the } x\text{-axis from } x = 2 \text{ to } x = 3)$
 $- (\text{area between } f(x) \text{ and the } x\text{-axis from } x = 3 \text{ to } x = 4)$
 $= 1.5 - 1.5 = 0$

d $\int_0^7 f(x) dx = (\text{area between } f(x) \text{ and the } x\text{-axis from } x = 0 \text{ to } x = 3)$
 $- (\text{area between } f(x) \text{ and the } x\text{-axis from } x = 3 \text{ to } x = 7)$
 $= 6.5 - 9 = -2.5$

6 a $\int_0^4 f(x) dx = \text{area of semi-circle with radius 2}$
 $= \frac{1}{2}\pi(2)^2 = 2\pi$

b $\int_4^6 f(x) dx = -(\text{area of 2 by 2 rectangle})$
 $= -(2 \times 2) = -4$

c $\int_6^8 f(x) dx = \text{area of semi-circle with radius 1}$
 $= \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

d $\int_0^8 f(x) dx = \int_0^4 f(x) dx + \int_4^6 f(x) dx + \int_6^8 f(x) dx$
 $= 2\pi + (-4) + \frac{\pi}{2} = \frac{5\pi}{2} - 4$

7 a $\int_2^4 f(x) dx + \int_4^7 f(x) dx$
 $= \int_2^7 f(x) dx$

8 a $\int_1^3 f(x) dx + \int_3^6 f(x) dx = \int_1^6 f(x) dx$
 $\therefore \int_3^6 f(x) dx = \int_1^6 f(x) dx - \int_1^3 f(x) dx$
 $= (-3) - 2$
 $= -5$

b $\int_0^2 f(x) dx + \int_2^4 f(x) dx + \int_4^6 f(x) dx = \int_0^6 f(x) dx$
 $\therefore \int_2^4 f(x) dx = \int_0^6 f(x) dx - \int_4^6 f(x) dx - \int_0^2 f(x) dx$
 $= (7) - (-2) - (5)$
 $= 4$

9 a $\int_1^{-1} f(x) dx = - \int_{-1}^1 f(x) dx$
 $= -(-4)$
 $= 4$

c $\int_{-1}^1 2f(x) dx = 2 \int_{-1}^1 f(x) dx$
 $= 2(-4)$
 $= -8$

b $\int_{-1}^1 (2 + f(x)) dx = \int_{-1}^1 2 dx + \int_{-1}^1 f(x) dx$
 $= [2x]_{-1}^1 + (-4)$
 $= (2 - (-2)) - 4$
 $= 0$

d $\int_{-1}^1 kf(x) dx = 7$
 $\therefore k \int_{-1}^1 f(x) dx = 7$
 $\therefore k(-4) = 7$
 $\therefore k = -\frac{7}{4}$

10 $\int_2^3 (g'(x) - 1) dx = \int_2^3 g'(x) dx + \int_2^3 -1 dx$
 $= [g(x)]_2^3 + [-x]_2^3$
 $= (g(3) - g(2)) + (-3 - (-2))$
 $= 5 - 4 - 1$
 $= 0$

REVIEW SET 21A

1 a $\int \frac{4}{\sqrt{x}} dx = 4 \int x^{-\frac{1}{2}} dx$
 $= 4 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 8\sqrt{x} + c$

b $\int \frac{3}{1-2x} dx = 3 \int \frac{1}{1-2x} dx$
 $= 3(\frac{1}{-2}) \ln(1-2x) + c, \quad 1-2x > 0$
 $= -\frac{3}{2} \ln(1-2x) + c, \quad x < \frac{1}{2}$

c $\int \sin(4x-5) dx = -\frac{1}{4} \cos(4x-5) + c$

d $\int e^{4-3x} dx = \frac{1}{-3} e^{4-3x} + c$
 $= -\frac{1}{3} e^{4-3x} + c$

2 a $\int_{-5}^{-1} \sqrt{1-3x} dx = \int_{-5}^{-1} (1-3x)^{\frac{1}{2}} dx$
 $= \left[\frac{1}{\frac{1}{3}} \times \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-5}^{-1}$
 $= -\frac{2}{9} \left[(1-3x)^{\frac{3}{2}} \right]_{-5}^{-1}$
 $= -\frac{2}{9} \left(4^{\frac{3}{2}} - 16^{\frac{3}{2}} \right)$
 $= -\frac{2}{9}(8 - 64) = 12^{\frac{4}{9}}$

b $\int_0^{\frac{\pi}{2}} \cos(\frac{x}{2}) dx = [2 \sin(\frac{x}{2})]_0^{\frac{\pi}{2}}$
 $= 2 \sin(\frac{\pi}{4}) - 2 \sin(0)$
 $= 2(\frac{1}{\sqrt{2}}) - 2(0)$
 $= \sqrt{2}$

3 $y = \sqrt{x^2 - 4} = (x^2 - 4)^{\frac{1}{2}}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}} \times 2x \\ &= \frac{x}{(x^2 - 4)^{\frac{1}{2}}} \\ &= \frac{x}{\sqrt{x^2 - 4}} \\ \therefore \int \frac{x}{\sqrt{x^2 - 4}} dx &= \sqrt{x^2 - 4} + c\end{aligned}$$

5 **a** $\int 4 \sin^2\left(\frac{x}{2}\right) dx$

$$\begin{aligned}&= \int 4\left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx \\ &= \int (2 - 2 \cos x) dx \\ &= 2x - 2 \sin x + c\end{aligned}$$

b $\int (2 - \cos x)^2 dx$

$$\begin{aligned}&= \int (4 - 4 \cos x + \cos^2 x) dx \\ &= \int (4 - 4 \cos x + \frac{1}{2} + \frac{1}{2} \cos 2x) dx \\ &= \frac{9}{2}x - 4 \sin x + \frac{1}{4} \sin 2x + c\end{aligned}$$

6 $\frac{d}{dx}((3x^2 + x)^3) = 3(3x^2 + x)^2 \times (6x + 1)$

$$\begin{aligned}\therefore \int 3(3x^2 + x)^2(6x + 1) dx &= (3x^2 + x)^3 + c_1 \\ \therefore 3 \int (3x^2 + x)^2(6x + 1) dx &= (3x^2 + x)^3 + c_1 \\ \therefore \int (3x^2 + x)^2(6x + 1) dx &= \frac{1}{3}(3x^2 + x)^3 + c\end{aligned}$$

7 **a** $\int_1^4 (f(x) + 1) dx = \int_1^4 f(x) dx + \int_1^4 1 dx$

$$\begin{aligned}&= 3 + [x]_1^4 \\ &= 3 + (4 - 1) = 6\end{aligned}$$

b $\int_1^2 f(x) dx - \int_4^2 f(x) dx = \int_1^2 f(x) dx + \int_2^4 f(x) dx$

$$\begin{aligned}&= \int_1^4 f(x) dx \\ &= 3\end{aligned}$$

8 $\int_0^a e^{1-2x} dx = \frac{e}{4}$

$$\therefore e^{1-2a} = \frac{e}{2}$$

$$\therefore \left[\frac{1}{-2}e^{1-2x}\right]_0^a = \frac{e}{4}$$

$$\therefore 1 - 2a = \ln\left(\frac{e}{2}\right) = \ln e - \ln 2$$

$$\therefore (-\frac{1}{2}e^{1-2a}) - (-\frac{1}{2}e^1) = \frac{e}{4}$$

$$\therefore -\frac{1}{2}e^{1-2a} + \frac{e}{2} = \frac{e}{4}$$

$$\therefore \frac{1}{2}e^{1-2a} = \frac{e}{4}$$

$$\therefore a = \ln 2^{\frac{1}{2}}$$

$$\therefore a = \ln \sqrt{2}$$

9 Given: $f''(x) = 2 \sin(2x)$, $f'(\frac{\pi}{2}) = 0$ and $f(0) = 3$

$$\begin{aligned}\text{Now } f'(x) &= \int 2 \sin(2x) dx \\ &= -\cos(2x) + c \\ \text{But } f'(\frac{\pi}{2}) &= 0, \text{ so } -\cos(\pi) + c = 0 \\ &\therefore -(-1) + c = 0 \\ &\therefore c = -1 \\ \therefore f'(x) &= -\cos(2x) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(x) &= \int (-\cos(2x) - 1) dx \\ &= -\frac{1}{2} \sin(2x) - x + k \\ \text{But } f(0) &= 3, \text{ so } -\frac{1}{2} \sin(0) - 0 + k = 3 \\ &\therefore k = 3 \\ \text{so } f(x) &= -\frac{1}{2} \sin(2x) - x + 3 \\ \therefore f(\frac{\pi}{2}) &= -\frac{1}{2} \sin(\pi) - \frac{\pi}{2} + 3 \\ &= 3 - \frac{\pi}{2}\end{aligned}$$

4 $\int_0^b \cos x dx = \frac{1}{\sqrt{2}}$, $0 < b < \pi$

$$\therefore [\sin x]_0^b = \frac{1}{\sqrt{2}}$$

$$\therefore \sin b - \sin 0 = \frac{1}{\sqrt{2}}$$

$$\therefore \sin b = \frac{1}{\sqrt{2}}$$

$$\therefore b = \frac{\pi}{4}, \frac{3\pi}{4} \quad \{0 < b < \pi\}$$

10 $\int_0^{\frac{\pi}{6}} \sin^2\left(\frac{x}{2}\right) dx$

$$\begin{aligned}&= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{2} \sin x\right]_0^{\frac{\pi}{6}} \\ &= \frac{\pi}{12} - \frac{1}{2}\left(\frac{1}{2}\right) - 0 + 0 \\ &= \frac{\pi}{12} - \frac{1}{4}\end{aligned}$$

REVIEW SET 21B

1 **a** $\frac{dy}{dx} = (x^2 - 1)^2$

$$\therefore y = \int (x^2 - 1)^2 dx$$

$$\begin{aligned}&= \int (x^4 - 2x^2 + 1) dx \\ &= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c\end{aligned}$$

b $\frac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$

$$\therefore y = \int (400 - 20e^{-\frac{x}{2}}) dx$$

$$\begin{aligned}&= 400x - \frac{20e^{-\frac{x}{2}}}{-\frac{1}{2}} + c \\ &= 400x + 40e^{-\frac{x}{2}} + c\end{aligned}$$

2 Using technology: **a** $\int_{-2}^0 4e^{-x^2} dx \approx 3.528$ **b** $\int_0^1 \frac{10x}{\sqrt{3x+1}} dx \approx 2.963$

3 $\frac{d}{dx}(\ln x)^2 = 2(\ln x)^1 \left(\frac{1}{x}\right)$

$$\begin{aligned}&= \frac{2 \ln x}{x} \\ \therefore \int \frac{2 \ln x}{x} dx &= (\ln x)^2 + c_1 \\ \therefore \int \frac{\ln x}{x} dx &= \frac{1}{2}(\ln x)^2 + c\end{aligned}$$

4 Given: $f''(x) = 18x + 10$, $f(0) = -1$, $f(1) = 13$

$$\begin{aligned}f'(x) &= \int (18x + 10) dx \\ &= 9x^2 + 10x + c\end{aligned}$$

$$\therefore f(x) = 3x^3 + 5x^2 + cx + d$$

But $f(0) = -1$ so $d = -1$

$$\therefore f(x) = 3x^3 + 5x^2 + cx - 1$$

And $f(1) = 13$ so $3 + 5 + c - 1 = 13$

$$\therefore c + 7 = 13$$

$$\therefore c = 6$$

$$\therefore f(x) = 3x^3 + 5x^2 + 6x - 1$$

5 Using technology: **a** $\int_3^4 \frac{x}{\sqrt{2x+1}} dx \approx 1.23617$ **b** $\int_0^1 x^2 e^{x+1} dx \approx 1.95249$

6 **a** $f''(x) = 3x^2 + 2x$

$$\begin{aligned}\therefore f'(x) &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c\end{aligned}$$

$$\therefore f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + d$$

But $f(0) = 3$ so $d = 3$

$$\therefore f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + 3$$

Also, $f(2) = 3$ so $4 + \frac{8}{3} + 2c + 3 = 3$

$$\therefore \frac{20}{3} = -2c$$

$$\therefore c = -\frac{10}{3}$$

$$\therefore f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{10}{3}x + 3$$

the normal has gradient $-\frac{3}{26}$

equation is $\frac{y-3}{x-2} = -\frac{3}{26}$

$$\therefore y - 3 = -\frac{3}{26}(x-2)$$

$$\therefore y = -\frac{3}{26}x + \frac{6}{26} + 3$$

or $3x + 26y = 84$

7 a $(e^x + 2)^3$
 $= (e^x)^3 + 3(e^x)^2(2) + 3(e^x)(2)^2 + (2)^3$
 $= e^{3x} + 6e^{2x} + 12e^x + 8$

c Using technology, $\int_0^1 (e^x + 2)^3 dx \approx 54.1$

b $\int_0^1 (e^x + 2)^3 dx$
 $= \left[\frac{1}{3}e^{3x} + 3e^{2x} + 12e^x + 8x \right]_0^1$
 $= \left(\frac{1}{3}e^3 + 3e^2 + 12e + 8 \right) - \left(\frac{1}{3} + 3 + 12 \right)$
 $= \frac{1}{3}e^3 + 3e^2 + 12e - 7\frac{1}{3} \quad (\approx 54.1)$

REVIEW SET 21C

1 a $\int \left(2e^{-x} - \frac{1}{x} + 3 \right) dx$
 $= -2e^{-x} - \ln x + 3x + c, \quad x > 0$

c $\int (3 + e^{2x-1})^2 dx$
 $= \int (9 + 6e^{2x-1} + e^{4x-2}) dx$
 $= 9x + 3e^{2x-1} + \frac{1}{4}e^{4x-2} + c$

2 $f'(x) = x^2 - 3x + 2$
 $\therefore f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$
But $f(1) = 3$
so $\frac{1}{3} - \frac{3}{2} + 2 + c = 3$
 $\therefore c = 1 - \frac{1}{3} + 1\frac{1}{2}$
 $\therefore c = 2\frac{1}{6}$
 $\therefore f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 2\frac{1}{6}$

b $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$
 $= \int \left(x - 2 + \frac{1}{x} \right) dx$
 $= \frac{1}{2}x^2 - 2x + \ln x + c, \quad x > 0$

3 $\int_2^3 \frac{1}{\sqrt{3x-4}} dx = \int_2^3 (3x-4)^{-\frac{1}{2}} dx$
 $= \left[\frac{\frac{1}{3}(3x-4)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^3$
 $= \left[\frac{2}{3}\sqrt{3x-4} \right]_2^3$
 $= \frac{2}{3}\sqrt{5} - \frac{2}{3}\sqrt{2}$
 $= \frac{2}{3}(\sqrt{5} - \sqrt{2})$

4 $\int_0^{\frac{\pi}{3}} \cos^2 \left(\frac{x}{2} \right) dx$
 $= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2} \cos x \right) dx$
 $= \left[\frac{1}{2}x + \frac{1}{2} \sin x \right]_0^{\frac{\pi}{3}}$
 $= \frac{\pi}{6} + \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right) - 0 - 0$
 $= \frac{\pi}{6} + \frac{\sqrt{3}}{4}$

5 $\frac{d}{dx} (e^{-2x} \sin x) = -2e^{-2x} \sin x + e^{-2x} \cos x \quad \{\text{product rule}\}$
 $= e^{-2x}(\cos x - 2 \sin x)$
 $\therefore \int_0^{\frac{\pi}{2}} e^{-2x}(\cos x - 2 \sin x) dx = \left[e^{-2x} \sin x \right]_0^{\frac{\pi}{2}}$
 $= e^{-\pi}(1) - e^0(0) = e^{-\pi}$

6 If $n \neq -1$, $\int (2x+3)^n dx = \frac{1}{2} \frac{(2x+3)^{n+1}}{n+1} + c = \frac{(2x+3)^{n+1}}{2(n+1)} + c$

If $n = -1$, $\int (2x+3)^{-1} dx = \int \frac{1}{2x+3} dx = \frac{1}{2} \ln(2x+3) + c, \quad 2x+3 > 0$

So, $\int (2x+3)^n dx = \begin{cases} \frac{(2x+3)^{n+1}}{2(n+1)} + c & \text{if } n \neq -1 \\ \frac{1}{2} \ln(2x+3) + c & \text{if } n = -1, \quad x > -\frac{3}{2} \end{cases}$

7 $f'(x) = 2\sqrt{x} + \frac{a}{\sqrt{x}}$
 $= 2x^{\frac{1}{2}} + ax^{-\frac{1}{2}}$
 $\therefore f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}} + c$
 $= \frac{4x\sqrt{x}}{3} + 2a\sqrt{x} + c$
Now $f(0) = 2$ so $c = 2$
 $\therefore f(x) = \frac{4x\sqrt{x}}{3} + 2a\sqrt{x} + 2$

Also, $f(1) = 4$ so $\frac{4}{3} + 2a + 2 = 4$
 $\therefore 2a = \frac{2}{3}$
 $\therefore a = \frac{1}{3}$
 $\therefore f'(x) = 2\sqrt{x} + \frac{1}{3\sqrt{x}} = \frac{6x+1}{3\sqrt{x}}$

Now $f(x)$ is only defined for $x > 0$,
so $f'(x) > 0$ for all x in the domain.
 \therefore the function has no stationary points.

8 $\int_a^{2a} (x^2 + ax + 2) dx = \frac{73a}{2}$
 $\therefore \left[\frac{x^3}{3} + \frac{ax^2}{2} + 2x \right]_a^{2a} = \frac{73a}{2}$
 $\therefore \left(\frac{8a^3}{3} + \frac{a}{2}(4a^2) + 4a \right) - \left(\frac{a^3}{3} + \frac{a^3}{2} + 2a \right) = \frac{73a}{2}$
 $\frac{8a^3}{3} + 2a^3 + 4a - \frac{a^3}{3} - \frac{a^3}{2} - 2a = \frac{73a}{2}$
 $\therefore 16a^3 + 12a^3 + 24a - 2a^3 - 3a^3 - 12a = 219a$
 $\therefore 23a^3 - 207a = 0$
 $\therefore 23a(a^2 - 9) = 0$
 $\therefore 23a(a+3)(a-3) = 0$
 $\therefore a = 0 \text{ or } a = \pm 3$