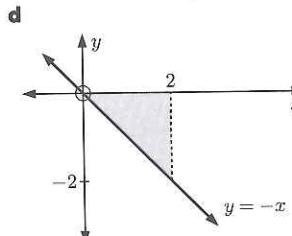
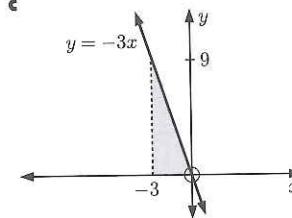
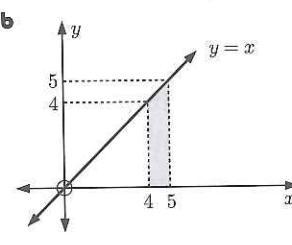
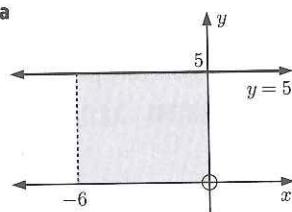


Chapter 22

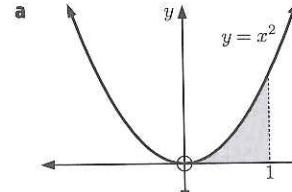
APPLICATIONS OF INTEGRATION

EXERCISE 22A.1

1



2



$$\text{Area} = \int_0^1 x^2 dx \\ = \left[\frac{x^3}{3} \right]_0^1 \\ = \frac{1}{3} - 0 \\ = \frac{1}{3} \text{ units}^2$$

i Area = $5 \times 6 = 30 \text{ units}^2$

ii Area = $\int_{-6}^0 5 dx = [5x]_{-6}^0 = 5(0) - 5(-6) = 30 \text{ units}^2$

i Area = $\left(\frac{4+5}{2}\right) \times 1 = \frac{9}{2} \text{ units}^2$

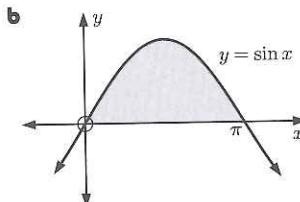
ii Area = $\int_4^5 x dx = \left[\frac{1}{2}x^2\right]_4^5 = \frac{1}{2}(25) - \frac{1}{2}(16) = \frac{9}{2} \text{ units}^2$

i Area = $\frac{1}{2} \times 3 \times 9 = \frac{27}{2} \text{ units}^2$

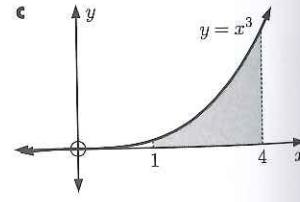
ii Area = $\int_{-3}^0 -3x dx = \left[-\frac{3}{2}x^2\right]_{-3}^0 = -\frac{3}{2}(0) - (-\frac{3}{2})(9) = \frac{27}{2} \text{ units}^2$

i Area = $\frac{1}{2} \times 2 \times 2 = 2 \text{ units}^2$

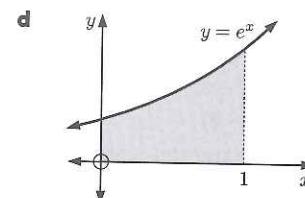
ii Area = $-\int_0^2 -x dx = -\left[-\frac{1}{2}x^2\right]_0^2 = -\left(-\frac{1}{2}(4) - (-\frac{1}{2})(0)\right) = 2 \text{ units}^2$



$$\text{Area} = \int_0^\pi \sin x dx \\ = [-\cos x]_0^\pi \\ = -\cos \pi - (-\cos 0) \\ = 2 \text{ units}^2$$



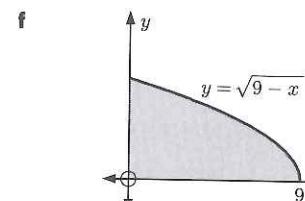
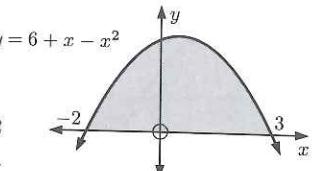
$$\text{Area} = \int_1^4 x^3 dx \\ = \left[\frac{x^4}{4} \right]_1^4 \\ = \frac{256}{4} - \frac{1}{4} \\ = 63\frac{3}{4} \text{ units}^2$$



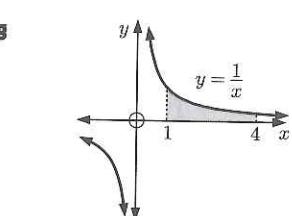
$$\text{Area} = \int_0^1 e^x dx \\ = [e^x]_0^1 \\ = e - 1 \\ \approx 1.72 \text{ units}^2$$

e The graph cuts the x -axis at $y = 0$.
 $\therefore 6 + x - x^2 = 0$
 $\therefore (3 - x)(2 + x) = 0$
 $\therefore x = 3 \text{ or } -2$
The x -intercepts are 3 and -2.

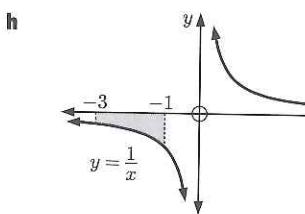
$$\text{Area} = \int_{-2}^3 (6 + x - x^2) dx \\ = \left[6x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^3 \\ = (18 + \frac{9}{2} - 9) - (-12 + 2 + \frac{8}{3}) \\ = 20\frac{5}{6} \text{ units}^2$$



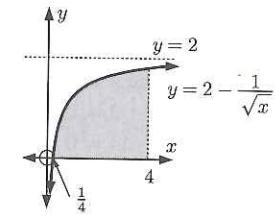
$$\text{Area} = \int_0^9 (9 - x)^{\frac{1}{2}} dx \\ = \left[\left(\frac{1}{-1} \right) \frac{(9 - x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^9 \\ = -\frac{2}{3} \left[(9 - x)^{\frac{3}{2}} \right]_0^9 \\ = -\frac{2}{3}(0 - 27) \\ = 18 \text{ units}^2$$



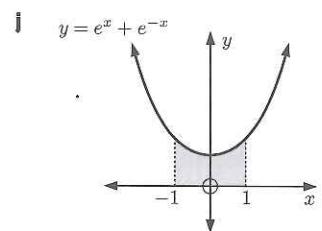
$$\text{Area} = \int_1^4 \frac{1}{x} dx \\ = [\ln x]_1^4 \quad \{x > 0\} \\ = \ln 4 - \ln 1 \\ = \ln 4 - 0 \\ \approx 1.39 \text{ units}^2$$



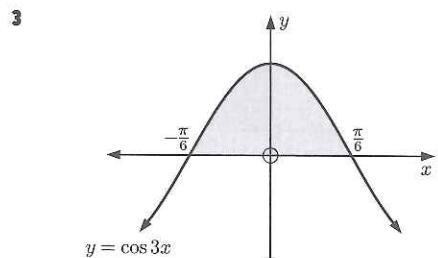
$$\text{Area} = - \int_{-3}^{-1} \frac{1}{x} dx \\ = \int_1^3 \frac{1}{x} dx \quad \{\text{by symmetry}\} \\ = [\ln x]_1^3 \quad \{x > 0\} \\ = \ln 3 - \ln 1 \\ = \ln 3 - 0 \\ \approx 1.10 \text{ units}^2$$



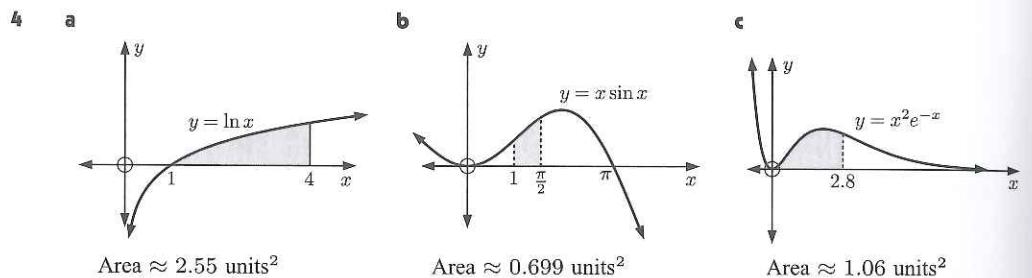
$$\text{Area} = \int_{\frac{1}{4}}^4 \left(2 - \frac{1}{\sqrt{x}}\right) dx \\ = \int_{\frac{1}{4}}^4 (2 - x^{-\frac{1}{2}}) dx \\ = \left[2x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_{\frac{1}{4}}^4 \\ = [2x - 2\sqrt{x}]_{\frac{1}{4}}^4 \\ = (8 - 4) - (\frac{1}{2} - 1) \\ = 4\frac{1}{2} \text{ units}^2$$



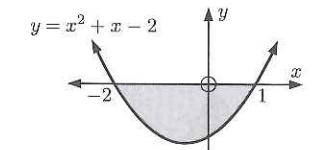
$$\begin{aligned} \text{Area} &= \int_{-1}^1 (e^x + e^{-x}) dx \\ &= [e^x - e^{-x}]_{-1}^1 \\ &= (e - e^{-1}) - (e^{-1} - e) \\ &= 2e - \frac{2}{e} \\ &\approx 4.70 \text{ units}^2 \end{aligned}$$



$$\begin{aligned} y = \cos 3x \text{ has zeros at } & \left\{ -\frac{\pi}{6}, \frac{\pi}{6} \right\} \\ \therefore \text{area} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 3x dx \\ &= \left[\frac{1}{3} \sin 3x \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\ &= \frac{1}{3} (\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})) \\ &= \frac{1}{3} (1 - (-1)) \\ &= \frac{2}{3} \text{ units}^2 \end{aligned}$$

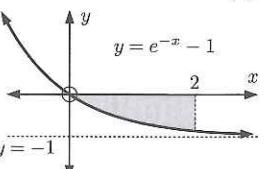
**EXERCISE 22A.2**

- 1 a** The curve cuts the x -axis when $y = 0$.
 $\therefore x^2 + x - 2 = 0$
 $\therefore (x+2)(x-1) = 0$
 $\therefore x = -2 \text{ or } 1$
 $\therefore \text{the } x\text{-intercepts are } -2 \text{ and } 1$



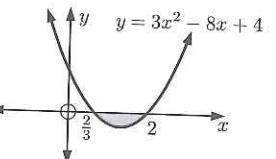
$$\begin{aligned} \text{Area} &= \int_{-2}^1 [0 - (x^2 + x - 2)] dx \\ &= \int_{-2}^1 (-x^2 - x + 2) dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

- b** The curve cuts the x -axis at $(0, 0)$.



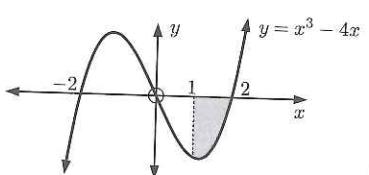
$$\begin{aligned} \text{Area} &= \int_0^2 [0 - (e^{-x} - 1)] dx \\ &= \int_0^2 (1 - e^{-x}) dx \\ &= [x + e^{-x}]_0^2 \\ &= (2 + e^{-2}) - (0 + e^0) \\ &= 1 + e^{-2} \text{ units}^2 \end{aligned}$$

- c** The curve cuts the x -axis when $y = 0$.
 $\therefore 3x^2 - 8x + 4 = 0$
 $\therefore (3x-2)(x-2) = 0$
 $\therefore x = 2 \text{ or } \frac{2}{3}$
 $\therefore \text{the } x\text{-intercepts are } 2 \text{ and } \frac{2}{3}$



$$\begin{aligned} \text{Area} &= \int_{\frac{2}{3}}^2 [0 - (3x^2 - 8x + 4)] dx \\ &= \int_{\frac{2}{3}}^2 (-3x^2 + 8x - 4) dx \\ &= \left[-x^3 + 4x^2 - 4x \right]_{\frac{2}{3}}^2 \\ &= (-8 + 16 - 8) - \left(-\frac{8}{27} + \frac{16}{9} - \frac{8}{3} \right) \\ &= 1\frac{5}{27} \text{ units}^2 \end{aligned}$$

- e** The curve cuts the x -axis when $y = 0$.
 $\therefore x^3 - 4x = 0$
 $\therefore x(x^2 - 4) = 0$
 $\therefore x(x+2)(x-2) = 0$
 $\therefore \text{the } x\text{-intercepts are } 0 \text{ and } \pm 2$



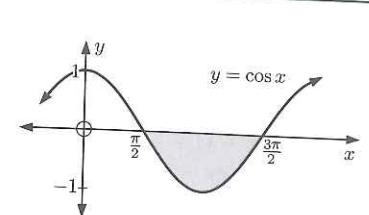
$$\begin{aligned} \text{Area} &= \int_1^2 [0 - (x^3 - 4x)] dx \\ &= \int_1^2 (-x^3 + 4x) dx \\ &= \left[-\frac{x^4}{4} + 2x^2 \right]_1^2 \\ &= (-4 + 8) - \left(-\frac{1}{4} + 2 \right) \\ &= 2\frac{1}{4} \text{ units}^2 \end{aligned}$$

- g** The curve cuts the x -axis when $y = 0$.

$$\begin{aligned} \therefore \sin^2 x &= 0 \\ \therefore \sin x &= 0 \\ \therefore x &= 0 + k\pi, \quad k \text{ an integer} \end{aligned}$$

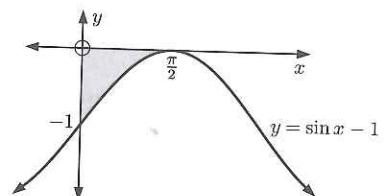
So, the first two non-negative x -intercepts are $0, \pi$.

$$\begin{aligned} \text{Area} &= \int_0^\pi [\sin^2 x - 0] dx \\ &= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{4} \sin 2x \right]_0^\pi \\ &= \left(\frac{1}{2}\pi - \frac{1}{4} \sin(2\pi) \right) - \left(\frac{1}{2}(0) - \frac{1}{4} \sin(0) \right) \\ &= \frac{\pi}{2} \text{ units}^2 \end{aligned}$$

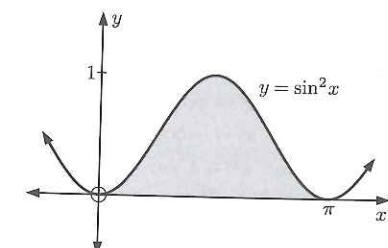


$$\begin{aligned} \text{Area} &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [0 - \cos x] dx \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x dx \\ &= [-\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= -\sin(\frac{3\pi}{2}) - (-\sin(\frac{\pi}{2})) \\ &= -(-1) - (-1) \\ &= 2 \text{ units}^2 \end{aligned}$$

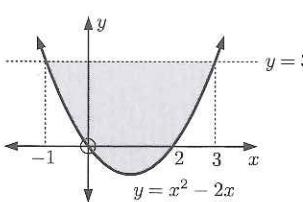
- f** $y = \sin x - 1$ is the graph of $\sin x$ translated vertically by -1 .



$$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{2}} [0 - (\sin x - 1)] dx \\ &= \int_0^{\frac{\pi}{2}} (1 - \sin x) dx \\ &= [x + \cos x]_0^{\frac{\pi}{2}} \\ &= (\frac{\pi}{2} + \cos \frac{\pi}{2}) - (0 + \cos 0) \\ &= \frac{\pi}{2} - 1 \text{ units}^2 \end{aligned}$$

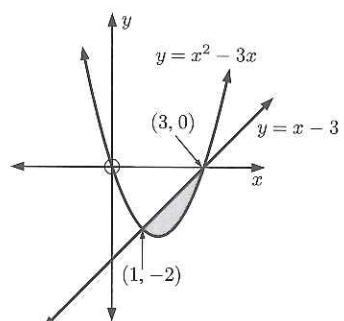


- 2 $y = x^2 - 2x$ meets $y = 3$
when $x^2 - 2x = 3$
 $\therefore x^2 - 2x - 3 = 0$
 $\therefore (x-3)(x+1) = 0$
 $\therefore x = 3 \text{ or } -1$



$$\begin{aligned} A &= \int_{-1}^3 [3 - (x^2 - 2x)] dx \\ &= \int_{-1}^3 (3 + 2x - x^2) dx \\ &= \left[3x + x^2 - \frac{x^3}{3} \right]_{-1}^3 \\ &= (9 + 9 - 9) - (-3 + 1 + \frac{1}{3}) \\ &= 10\frac{2}{3} \text{ units}^2 \end{aligned}$$

3



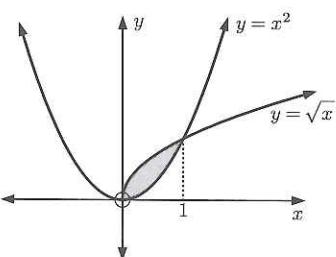
- a The graphs meet where $x - 3 = x^2 - 3x$
 $\therefore x^2 - 4x + 3 = 0$
 $\therefore (x-1)(x-3) = 0$
 $\therefore x = 1 \text{ or } 3$

b The graphs meet at $(1, -2)$ and $(3, 0)$

$$\begin{aligned} c \text{ Area} &= \int_1^3 [(x-3) - (x^2 - 3x)] dx \\ &= \int_1^3 (-3 + 4x - x^2) dx \\ &= \left[-3x + 2x^2 - \frac{x^3}{3} \right]_1^3 \\ &= (-9 + 18 - 9) - (-3 + 2 - \frac{1}{3}) \\ &= 1\frac{1}{3} \text{ units}^2 \end{aligned}$$

- 4 $y = \sqrt{x}$ meets $y = x^2$ where $\sqrt{x} = x^2$
 $\therefore x = x^4$
 $\therefore x^4 - x = 0$
 $\therefore x(x^3 - 1) = 0$
 $\therefore x(x-1)(x^2+x+1) = 0$
 $\therefore x = 0 \text{ or } 1$

The factor (x^2+x+1) has no real root since $\Delta = -3$ which is < 0 .



$$\begin{aligned} \text{Area} &= \int_0^1 (\sqrt{x} - x^2) dx \\ &= \int_0^1 (x^{\frac{1}{2}} - x^2) dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \text{ unit}^2 \end{aligned}$$

- 5 a $y = e^x - 1$ has no vertical asymptotes.

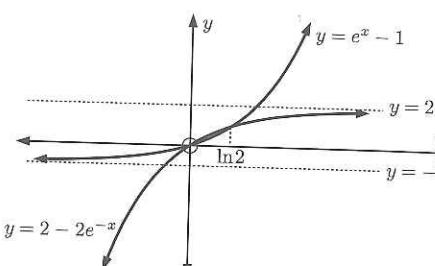
As $x \rightarrow \infty$, $e^x \rightarrow \infty$
As $x \rightarrow -\infty$, $e^x \rightarrow 0$
so $e^x - 1 \rightarrow -1$ (above)
 $\therefore y = -1$ is a horizontal asymptote.
 $y = 0$ when $e^x - 1 = 0$
 $\therefore e^x = 1$
 $\therefore x = 0$
 $\therefore x\text{-intercept is } (0, 0).$

This is also the $y\text{-intercept}$.

- b $y = 2 - 2e^{-x}$ has no vertical asymptotes.

As $x \rightarrow \infty$, $e^{-x} \rightarrow 0$
so $2 - 2e^{-x} \rightarrow 2$ (below)
 $\therefore y = 2$ is a horizontal asymptote.
 $y = 0$ when $2 - 2e^{-x} = 0$
 $\therefore e^{-x} = 1$
 $\therefore x = 0$
 $\therefore x\text{-intercept is } (0, 0).$

This is also the $y\text{-intercept}$.



- b $y = e^x - 1$ meets $y = 2 - 2e^{-x}$

where $e^x - 1 = 2 - 2e^{-x}$

$$\therefore e^{2x} - e^x = 2e^x - 2 \quad \{ \times e^x \}$$

$$\therefore e^{2x} - 3e^x + 2 = 0$$

$$\therefore (e^x - 1)(e^x - 2) = 0$$

$$\therefore e^x = 1 \text{ or } 2$$

$$\therefore x = 0 \text{ or } \ln 2$$

c The graphs meet at $(0, 0)$ and $(\ln 2, 1)$.

$$\begin{aligned} c \text{ Area} &= \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)] dx \\ &= \int_0^{\ln 2} (3 - e^x - 2e^{-x}) dx \\ &= \left[3x - e^x + 2e^{-x} \right]_0^{\ln 2} \\ &= (3 \ln 2 - 2 + 1) - (0 - 1 + 2) \\ &= 3 \ln 2 - 2 \\ &\approx 0.0794 \text{ units}^2 \end{aligned}$$

- 6 $y = 2e^x$ meets $y = e^{2x}$ where

$$2e^x = e^{2x}$$

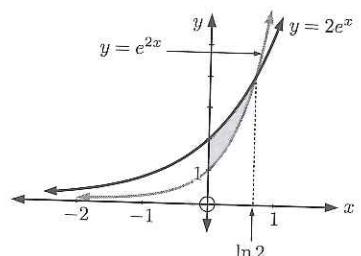
$$\therefore e^{2x} - 2e^x = 0$$

$$\therefore e^x(e^x - 2) = 0$$

$$\therefore e^x = 2 \quad \{ e^x > 0 \text{ for all } x \}$$

$$\therefore x = \ln 2$$

$$\begin{aligned} \text{Area} &= \int_0^{\ln 2} (2e^x - e^{2x}) dx \\ &= \left[2e^x - \frac{1}{2}e^{2x} \right]_0^{\ln 2} \\ &= (4 - 2) - (2 - \frac{1}{2}) \\ &= \frac{1}{2} \text{ unit}^2 \end{aligned}$$



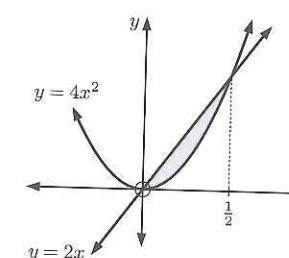
- 7 $y = 2x$ meets $y = 4x^2$ where

$$2x = 4x^2$$

$$\therefore 4x^2 - 2x = 0$$

$$\therefore 2x(2x - 1) = 0$$

$$\therefore x = 0 \text{ or } \frac{1}{2}$$



$$\begin{aligned} \text{Area} &= \int_0^{\frac{1}{2}} (2x - 4x^2) dx \\ &= \left[x^2 - \frac{4}{3}x^3 \right]_0^{\frac{1}{2}} \\ &= \left(\frac{1}{4} - \frac{4}{3} \left(\frac{1}{8} \right) \right) - (0 - 0) \\ &= \frac{1}{12} \text{ unit}^2 \end{aligned}$$

- 8 a Now $x^2 + y^2 = 9 \quad \therefore y^2 = 9 - x^2$

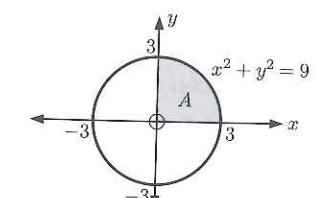
$$\therefore y = \pm\sqrt{9 - x^2}$$

In the upper half of the circle all y -values are ≥ 0
 $\therefore y = +\sqrt{9 - x^2}$ is the required equation.

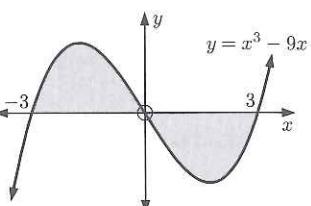
- b The shaded area is A where $A = \int_0^3 \sqrt{9 - x^2} dx$

This is a quarter of the area of a circle with radius 3 units.

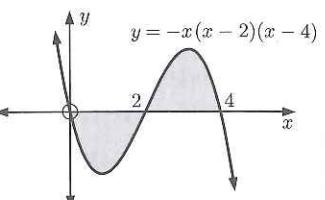
$$\therefore A = \frac{1}{4}(\pi \times 3^2) = \frac{9}{4}\pi \approx 7.07 \text{ units}^2$$



9 **a** $f(x) = x^3 - 9x$
 $= x(x^2 - 9)$
 $\therefore x(x+3)(x-3)$
 $\therefore y = f(x)$ cuts the x -axis at $0, \pm 3$
Area $= \int_{-3}^0 (x^3 - 9x) dx + \int_0^3 [0 - (x^3 - 9x)] dx$
 $= \left[\frac{x^4}{4} - \frac{9x^2}{2} \right]_0^3 + \left[-\frac{x^4}{4} + \frac{9x^2}{2} \right]_0^3$
 $= 0 - (\frac{81}{4} - \frac{81}{2}) + (-\frac{81}{4} + \frac{81}{2}) - 0$
 $= 40\frac{1}{2}$ units²

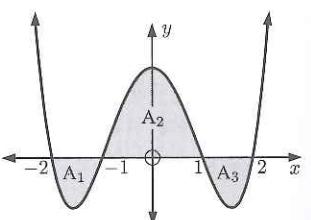


b $f(x) = -x(x-2)(x-4)$
 $= -x^3 + 6x^2 - 8x$
 $\therefore y = f(x)$ cuts the x -axis at $0, 2$ and 4



Area $= \int_0^2 [0 - (-x^3 + 6x^2 - 8x)] dx$
 $+ \int_2^4 (-x^3 + 6x^2 - 8x) dx$
 $= \int_0^2 (x^3 - 6x^2 + 8x) dx + \int_2^4 (-x^3 + 6x^2 - 8x) dx$
 $= \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 + \left[-\frac{x^4}{4} + 2x^3 - 4x^2 \right]_2^4$
 $= ([4 - 16 + 16] - 0) + ([-64 + 128 - 64] - [-4 + 16 - 16])$
 $= 8$ units²

c $f(x) = x^4 - 5x^2 + 4$
 $= (x^2 - 1)(x^2 - 4)$
 $= (x+1)(x-1)(x+2)(x-2)$
 $\therefore y = f(x)$ cuts the x -axis at $\pm 1, \pm 2$



$A_1 = \int_{-2}^{-1} [0 - (x^4 - 5x^2 + 4)] dx$
 $= \int_{-2}^{-1} (-x^4 + 5x^2 - 4) dx$
 $= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_{-2}^{-1}$
 $= (\frac{1}{5} - \frac{5}{3} + 4) - (-\frac{32}{5} - \frac{40}{3} + 8)$
 $= \frac{22}{15}$ units²

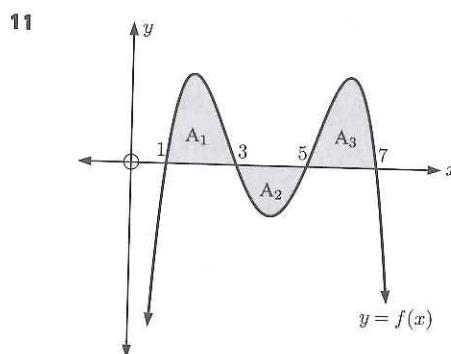
By symmetry, $A_3 = A_1 \quad \therefore$ area $= \frac{22}{15} + \frac{76}{15} + \frac{22}{15} = \frac{120}{15} = 8$ units²

10 **a** $y = \sin(2x)$ is the curve C_1 and $y = \sin x$ is the curve C_2 .

b The curves meet when $\sin(2x) = \sin x \quad \therefore x = 0 + k\pi$ or $x = \left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\} + 2k\pi, k$ an integer
 $\therefore 2\sin x \cos x - \sin x = 0$
 $\therefore \sin x(2\cos x - 1) = 0$
 $\therefore \sin x = 0$ or $\cos x = \frac{1}{2}$
 $\therefore A$ is at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

the x -coordinate of $A = \frac{\pi}{3}$
{smallest positive solution}

c Area $= \int_0^{\frac{\pi}{3}} (\sin(2x) - \sin x) dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin(2x)) dx$
 $= \left[-\frac{1}{2} \cos(2x) + \cos x \right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2} \cos(2x) \right]_{\frac{\pi}{3}}^{\pi}$
 $= \left(-\frac{1}{2} \cos \frac{2\pi}{3} + \cos \frac{\pi}{3} \right) - \left(-\frac{1}{2} \cos 0 + \cos 0 \right) + \left(-\cos \pi + \frac{1}{2} \cos 2\pi \right)$
 $- \left(-\cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} \right)$
 $= (\frac{1}{4} + \frac{1}{2}) - (-\frac{1}{2} + 1) + (1 + \frac{1}{2}) - (-\frac{1}{2} - \frac{1}{4})$
 $= 2\frac{1}{2}$ units²



a $\int_1^7 f(x) dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leq x \leq 7$. But $f(x)$ is not positive for $3 \leq x \leq 5$, so $\int_1^7 f(x) dx = A_1 - A_2 + A_3$ which is not the shaded area.

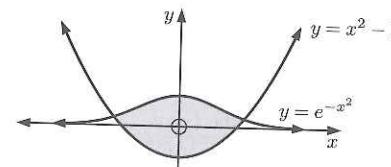
b shaded area
 $= \int_1^3 f(x) dx + \int_3^5 [0 - f(x)] dx + \int_5^7 f(x) dx$
 $= \int_1^3 f(x) dx - \int_3^5 f(x) dx + \int_5^7 f(x) dx$

12 **a** $y = \cos(2x)$ is the curve C_2 and $y = \cos^2 x$ is the curve C_1 .

b Point A lies on $y = \cos(2x)$. When $x = 0, y = \cos 0 = 1$. $\therefore A$ is at $(0, 1)$.
Point B lies on $y = \cos(2x)$. When $x = \frac{\pi}{4}, y = \cos \frac{\pi}{2} = 0$. $\therefore B$ is at $(\frac{\pi}{4}, 0)$.
Point C lies on $y = \cos^2 x$. When $x = \frac{\pi}{2}, y = \cos^2 \frac{\pi}{2} = 0$. $\therefore C$ is at $(\frac{\pi}{2}, 0)$.
Point D lies on $y = \cos(2x)$. When $x = \frac{3\pi}{4}, y = \cos \frac{3\pi}{2} = 0$. $\therefore D$ is at $(\frac{3\pi}{4}, 0)$.
Point E lies where the curves meet. Now $\cos(2\pi) = \cos^2 \pi = 1$. $\therefore E$ is at $(\pi, 1)$.

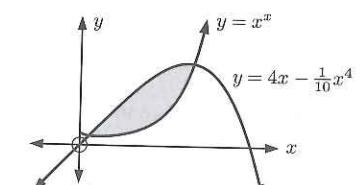
c $A = \int_0^{\pi} (\cos^2 x - \cos(2x)) dx$
 $= \int_0^{\pi} (\frac{1}{2} + \frac{1}{2} \cos(2x) - \cos(2x)) dx$
 $= \int_0^{\pi} (\frac{1}{2} - \frac{1}{2} \cos(2x)) dx$
 $= \left[\frac{x}{2} - \frac{1}{4} \sin(2x) \right]_0^{\pi} = (\frac{\pi}{2} - 0) - (0 - 0) = \frac{\pi}{2}$ units²

13 **a** The graphs meet when $e^{-x^2} = x^2 - 1$
 $\therefore x = \pm 1.1307$ {technology}



\therefore area $= \int_{-1.1307}^{1.1307} [e^{-x^2} - (x^2 - 1)] dx$
 ≈ 2.88 units² {technology}

b The graphs meet when $x^x = 4x - \frac{1}{10}x^4$
 $\therefore x \approx 0.1832$ or 2.2696 {technology}



\therefore area $= \int_{0.1832}^{2.2696} (4x - \frac{1}{10}x^4 - x^x) dx$
 ≈ 4.97 units² {technology}

14 Area = $\int_1^k \frac{1}{1+2x} dx = 0.2$ units²

$$\therefore \left[\frac{1}{2} \ln(1+2x) \right]_1^k = 0.2, \quad 1+2x > 0$$

$$\therefore [\ln(1+2x)]_1^k = 0.4$$

$$\therefore \ln(1+2k) - \ln 3 = 0.4$$

{since $k \geq 1$, $1+2x > 0$ for all x in the shaded region}

$$\therefore \ln\left(\frac{1+2k}{3}\right) = 0.4$$

$$\therefore \frac{1+2k}{3} = e^{0.4}$$

$$\therefore 1+2k = 3e^{0.4}$$

$$\therefore k = \frac{3e^{0.4} - 1}{2} \approx 1.7377$$

16 By symmetry, the area bounded by $x = 0$ and $x = a$ is $\frac{1}{2}(6a)$ units².

$$\therefore \int_0^a (x^2 + 2) dx = 3a$$

$$\therefore \left[\frac{x^3}{3} + 2x \right]_0^a = 3a$$

$$\therefore \frac{a^3}{3} + 2a - 0 = 3a$$

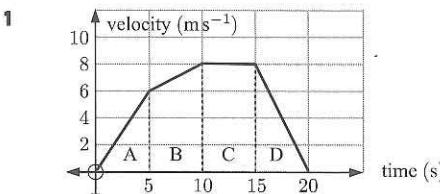
$$\therefore a^3 + 6a = 9a$$

$$\therefore a^3 - 3a = 0$$

$$\therefore a(a^2 - 3) = 0$$

$$\therefore a = 0 \text{ or } \pm \sqrt{3} \quad \therefore a = \sqrt{3} \quad \{\text{as } a > 0\}$$

EXERCISE 22B.1



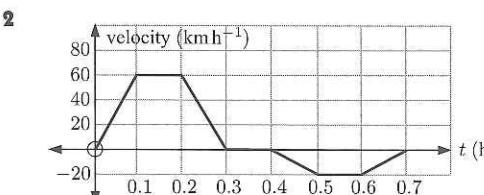
Total distance travelled

$$= \text{area A} + \text{area B} + \text{area C} + \text{area D}$$

$$= \frac{1}{2}(5 \times 6) + \left(\frac{6+8}{2} \right) 5 + 5 \times 8 + \frac{1}{2}(5 \times 8)$$

$$= 15 + 35 + 40 + 20$$

$$= 110 \text{ m}$$

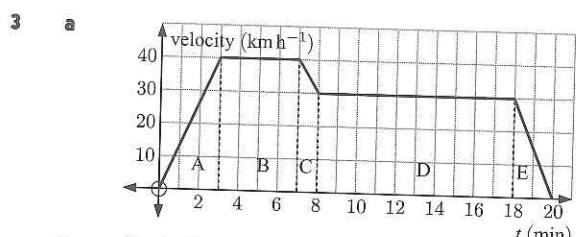


b Total distance travelled = area above the t -axis + area below the t -axis

$$= \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2} \right) 60 + \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2} \right) 20 \\ = 12 + 4 \\ = 16 \text{ km}$$

c Final displacement = area above the t -axis – area below the t -axis

$$= 12 - 4 \\ = 8 \text{ km from the starting point in the positive direction}$$



b Total distance travelled

$$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E} \\ = \frac{1}{60} \left[\frac{1}{2}(3 \times 40) + (40 \times 4) + \left(\frac{40+30}{2} \right) 1 + (10 \times 30) + \frac{1}{2}(2 \times 30) \right]$$

{the factor $\frac{1}{60}$ accounts for the fact that the times are in minutes while the speeds are in km h^{-1} }

$$= \frac{1}{60}[60 + 160 + 35 + 300 + 30]$$

$$= 9.75 \text{ km}$$

EXERCISE 22B.2

1 a $v(t) = s'(t) = 1 - 2t \text{ cm s}^{-1}, \quad t \geq 0$

which has sign diagram:
 0 $\begin{matrix} + \\ | \\ - \end{matrix}$ $\frac{1}{2}$

\therefore a direction reversal occurs at $t = \frac{1}{2}$.

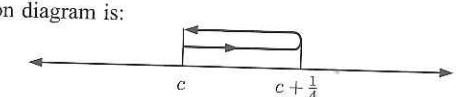
Now $s(t) = \int (1 - 2t) dt = t - \frac{2t^2}{2} + c = t - t^2 + c$

$$\therefore s(0) = c \quad \text{and motion diagram is:}$$

$$\text{and } s\left(\frac{1}{2}\right) = \frac{1}{4} + c$$

$$\text{and } s(1) = c$$

$$\therefore \text{total distance travelled} = (c + \frac{1}{4} - c) + (c + \frac{1}{4} - c) \\ = \frac{1}{2} \text{ cm}$$



b Displacement = $s(1) - s(0)$

$$= c - c$$

$$= 0 \text{ cm}$$

2 a $v(t) = s'(t) = t^2 - t - 2 \text{ cm s}^{-1}, \quad t \geq 0$
 $= (t-2)(t+1)$

which has sign diagram:
 0 $\begin{matrix} - \\ | \\ + \end{matrix}$ 2

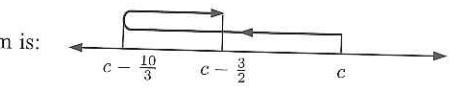
\therefore a direction reversal occurs at $t = 2$.

Now $s(t) = \int (t^2 - t - 2) dt = \frac{t^3}{3} - \frac{t^2}{2} - 2t + c$

$$s(0) = c$$

$$s(2) = c - \frac{10}{3}$$

$$s(3) = c - \frac{3}{2} \quad \therefore \text{motion diagram is:}$$



$$\therefore \text{total distance travelled} = \left(c - [c - \frac{10}{3}] \right) + \left(c - \frac{3}{2} - [c - \frac{10}{3}] \right) \\ = \frac{10}{3} - \frac{3}{2} + \frac{10}{3} \\ = \frac{31}{6} = 5\frac{1}{6} \text{ cm}$$

b Displacement = $s(3) - s(0)$

$$= c - \frac{3}{2} - c$$

$$= -\frac{3}{2} \text{ cm which is } 1\frac{1}{2} \text{ cm left of its starting point.}$$

3 $x'(t) = 16t - 4t^3$ units s^{-1} , $t \geq 0$
 $= 4t(4 - t^2)$
 $= 4t(2+t)(2-t)$ which has sign diagram:
 \therefore a direction reversal occurs at $t = 2$.

Now $x(t) = \int (16t - 4t^3) dt = 8t^2 - t^4 + c$

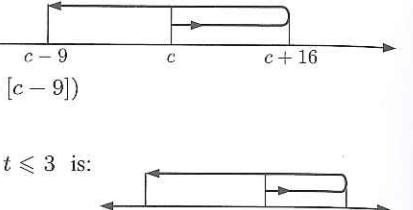
a $x(0) = c$ \therefore motion diagram for $0 \leq t \leq 3$ is:

$$x(2) = 32 - 16 + c = c + 16$$

$$x(3) = 72 - 81 + c = c - 9$$

$$\therefore \text{total distance travelled} = (c + 16 - c) + (c + 16 - [c - 9]) \\ = 41 \text{ units}$$

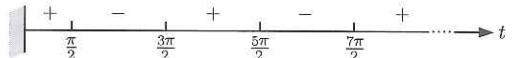
b $x(1) = 7 + c = c + 7$ \therefore motion diagram for $1 \leq t \leq 3$ is:



$$\therefore \text{total distance travelled} = (c + 16 - [c + 7]) + (c + 16 - [c - 9]) \\ = 34 \text{ units}$$

4 a $v(t) = \cos t \text{ ms}^{-1}$, $t \geq 0$

$\therefore v(t)$ has sign diagram:

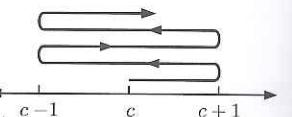


\therefore a direction reversal occurs at $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$

$$s(t) = \int \cos t dt = \sin t + c$$

$$\therefore s(0) = c$$

The motion diagram is:



\therefore the particle oscillates between the points $(c-1)$ and $(c+1)$.

b distance $= (c+1) - (c-1)$
 $= 2 \text{ units}$

5 $v(t) = 50 - 10e^{-0.5t} \text{ ms}^{-1}$, $t \geq 0$

a $v(0) = 50 - \frac{10}{e^0} = 50 - 10 = 40 \text{ ms}^{-1}$

c The velocity reaches 45 ms^{-1} when $45 = 50 - 10e^{-0.5t}$

$$\therefore 10e^{-\frac{t}{2}} = 5$$

$$\therefore e^{\frac{t}{2}} = 2$$

$$\therefore \frac{t}{2} = \ln 2$$

$$\therefore t = 2 \ln 2 \approx 1.39 \text{ seconds}$$

e $a(t) = v'(t)$
 $= -10e^{-0.5t}(-0.5)$
 $\therefore 5e^{-0.5t} \text{ ms}^{-2}$
 $= \frac{5}{e^{0.5t}} \text{ ms}^{-2}$

b $v(3) = 50 - \frac{10}{e^{1.5}} \approx 47.8 \text{ ms}^{-1}$

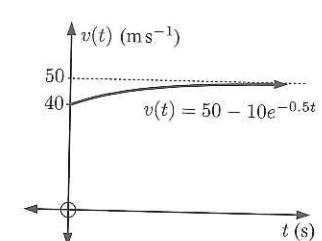
d $v(t) = 50 - \frac{10}{e^{\frac{t}{2}}}$

As $t \rightarrow \infty$, $\frac{10}{e^{\frac{t}{2}}} \rightarrow 0$ (from above)

$$\therefore v(t) \rightarrow 50 \text{ ms}^{-1} \text{ (below)}$$

$\therefore a(t) > 0$ for all t { $e^x > 0$ for all x }

\therefore the acceleration is always positive



g total distance travelled
 $= \int_0^3 (50 - 10e^{-0.5t}) dt$
 $= [50t + 20e^{-0.5t}]_0^3$
 $= 150 + 20e^{-1.5} - 20$
 $\approx 134 \text{ m}$

6 $a(t) = \frac{t}{10} - 3 \text{ ms}^{-2}$

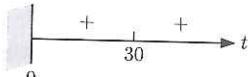
$$\therefore v(t) = \int \left(\frac{t}{10} - 3 \right) dt$$

$$= \frac{t^2}{20} - 3t + c$$

But $v(0) = 45 \therefore c = 45$

Now $v(t) = \frac{t^2}{20} - 3t + 45$
 $= \frac{t^2 - 60t + 900}{20}$
 $= \frac{(t-30)^2}{20}$

Sign diagram of $v(t)$:



The total distance travelled in the first minute

$$= \int_0^{60} \left(\frac{(t-30)^2}{20} \right) dt \quad \{ \text{as } (t-30)^2 \geq 0 \}$$

$$= \frac{1}{20} \int_0^{60} (t-30)^2 dt$$

$$= \frac{1}{20} \left[\frac{(t-30)^3}{3} \right]_0^{60}$$

$$= \frac{1}{60} ((30)^3 - (-30)^3)$$

$$= \frac{1}{60} (30^3 + 30^3)$$

$$= 900 \text{ m}$$

7 $a(t) = 4e^{-\frac{t}{20}} \text{ ms}^{-2}$

$$\therefore v(t) = \int 4e^{-\frac{t}{20}} dt$$

$$= 4 \cdot \frac{1}{-\frac{1}{20}} e^{-\frac{t}{20}} + c$$

$$= -80e^{-\frac{t}{20}} + c$$

Now $v(0) = 20 \text{ ms}^{-1}$

$\therefore c = 100$

$$\therefore v(t) = 100 - 80e^{-\frac{t}{20}}$$

a As $t \rightarrow \infty$, $e^{-\frac{t}{20}} \rightarrow 0$ (above) $\therefore v(t) \rightarrow 100$ (below)
 \therefore the object approaches a limiting velocity of 100 ms^{-1}

b The total distance travelled $= \int_0^{10} (100 - 80e^{-\frac{t}{20}}) dt \quad \{ v(t) > 0 \text{ for } 0 \leq t \leq 10 \}$
 $= [100t + 1600e^{-\frac{t}{20}}]_0^{10}$
 $= (1000 + 1600e^{-\frac{1}{2}}) - (0 + 1600)$
 $\approx 370 \text{ m}$

EXERCISE 22C

1 The marginal cost is $C'(x) = 3.15 + 0.004x$ € per gadget

$$\therefore C(x) = \int (3.15 + 0.004x) dx$$

$$= 3.15x + 0.002x^2 + c$$

But $C(0) = 450$ so $c = 450$

$$\therefore C(x) = 3.15x + 0.002x^2 + 450 \text{ euros}$$

$$\therefore C(800) = 3.15(800) + 0.002(800)^2 + 450$$

$$= €4250$$

So, the total cost is €4250.

- 2 a The marginal profit is $P'(x) = 15 - 0.03x$ dollars per plate

$$\therefore P(x) = \int (15 - 0.03x) dx \\ = 15x - 0.015x^2 + c$$

But $P(0) = -650$ so $c = -650$

$$\therefore P(x) = 15x - 0.015x^2 - 650 \text{ dollars}$$

- b The maximum profit occurs when $P'(x) = 0$, which is when $15 - 0.03x = 0$

$$\therefore 0.03x = 15 \\ \therefore x = \frac{15}{0.03} \\ \therefore x = 500$$

Now $P''(x) = -0.03 < 0 \therefore$ the profit is at a maximum when $x = 500$.

$$\text{The maximum profit} = P(500) = 15(500) - 0.015(500)^2 - 650 \\ = \$3100$$

- c In order for a profit to be made, $P(x)$ must be greater than 0

$$\therefore 15x - 0.015x^2 - 650 > 0$$

Using technology, the x -intercepts of $P(x)$ are $x_1 = 45.39$ and $x_2 = 954.6$

Since we cannot produce part plates, a profit is made for $46 \leq x \leq 954$.

- 3 $E'(t) = 350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)$ calories per day

$$\begin{aligned} \text{Total energy needs over the first week} &= \int_0^7 E'(t) dt \\ &= \int_0^7 [350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)] dt \\ &= \left[\frac{1}{0.15} \times \frac{350(80 + 0.15t)^{1.8}}{1.8} - 9600t - 9t^2 \right]_0^7 \\ &\approx 14400 \text{ calories} \end{aligned}$$

$$4 \quad \frac{dT}{dx} = \frac{-20}{x^{0.63}} = -20x^{-0.63}$$

$$\therefore T = \int -20x^{-0.63} dx \\ = \frac{-20x^{0.37}}{0.37} + c$$

Now when $x = 3$, $T = 100$

$$\therefore \frac{-20(3^{0.37})}{0.37} + c = 100$$

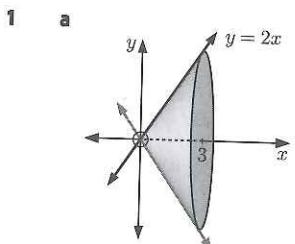
$$\therefore c = 100 + \frac{20(3^{0.37})}{0.37} \approx 181.1639$$

$$\therefore T \approx \frac{-20x^{0.37}}{0.37} + 181.1639$$

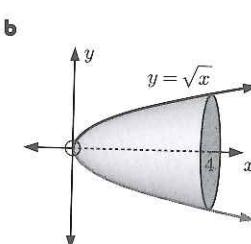
So, when $x = 6$, $T \approx -104.8925 + 181.1639 \approx 76.27$

\therefore the outer surface temperature is about 76.3°C

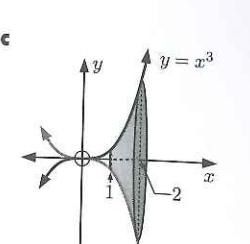
EXERCISE 22D.1



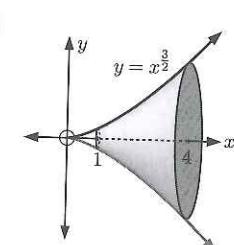
$$\begin{aligned} \text{Volume} &= \pi \int_0^3 (2x)^2 dx \\ &= 4\pi \int_0^3 x^2 dx \\ &= 4\pi \left[\frac{1}{3}x^3 \right]_0^3 \\ &= 4\pi(9 - 0) \\ &= 36\pi \text{ units}^3 \end{aligned}$$



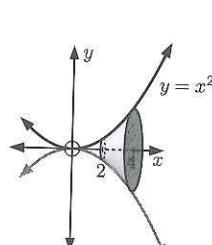
$$\begin{aligned} \text{Volume} &= \pi \int_0^4 (\sqrt{x})^2 dx \\ &= \pi \int_0^4 x dx \\ &= \pi \left[\frac{1}{2}x^2 \right]_0^4 \\ &= \pi(8 - 0) \\ &= 8\pi \text{ units}^3 \end{aligned}$$



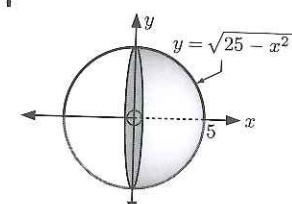
$$\begin{aligned} \text{Volume} &= \pi \int_1^2 (x^3)^2 dx \\ &= \pi \int_1^2 x^6 dx \\ &= \pi \left[\frac{1}{7}x^7 \right]_1^2 \\ &= \pi \left(\frac{128}{7} - \frac{1}{7} \right) \\ &= \frac{127\pi}{7} \text{ units}^3 \end{aligned}$$



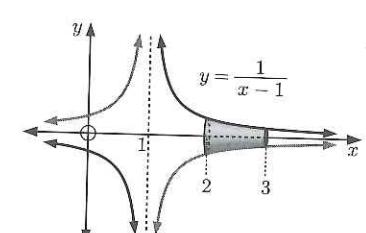
$$\begin{aligned} \text{Volume} &= \pi \int_1^4 (x^{\frac{3}{2}})^2 dx \\ &= \pi \int_1^4 x^3 dx \\ &= \pi \left[\frac{1}{4}x^4 \right]_1^4 \\ &= \pi \left(\frac{256}{4} - \frac{1}{4} \right) \\ &= \frac{255\pi}{4} \text{ units}^3 \end{aligned}$$



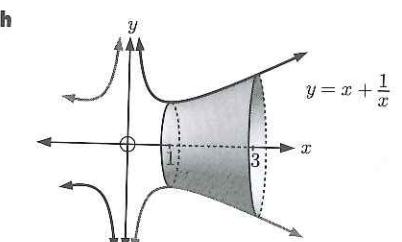
$$\begin{aligned} \text{Volume} &= \pi \int_2^4 (x^2)^2 dx \\ &= \pi \int_2^4 x^4 dx \\ &= \pi \left[\frac{1}{5}x^5 \right]_2^4 \\ &= \pi \left(\frac{1024}{5} - \frac{32}{5} \right) \\ &= \frac{992\pi}{5} \text{ units}^3 \end{aligned}$$



$$\begin{aligned} \text{Volume} &= \pi \int_0^5 (25 - x^2)^2 dx \\ &= \pi \int_0^5 25x^2 - x^4 dx \\ &= \pi \left[25x^3 - \frac{1}{5}x^5 \right]_0^5 \\ &= \pi \left(125 - \frac{125}{5} \right) \\ &= \pi \left(\frac{2}{3} \right) 125 \\ &= \frac{250\pi}{3} \text{ units}^3 \end{aligned}$$



$$\begin{aligned} \text{Volume} &= \pi \int_2^3 \left(\frac{1}{x-1} \right)^2 dx \\ &= \pi \int_2^3 (x-1)^{-2} dx \\ &= \pi \left[-\frac{1}{x-1} \right]_2^3 \\ &= \pi \left(-\frac{1}{2} + 1 \right) \\ &= \frac{\pi}{2} \text{ units}^3 \end{aligned}$$



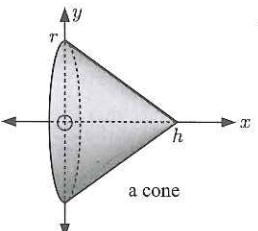
$$\begin{aligned} \text{Volume} &= \pi \int_1^3 \left(x + \frac{1}{x} \right)^2 dx \\ &= \pi \int_1^3 (x^2 + 2 + x^{-2}) dx \\ &= \pi \left[\frac{x^3}{3} + 2x - \frac{1}{x} \right]_1^3 \\ &= \pi \left[9 + 6 - \frac{1}{3} - \left(\frac{1}{3} + 2 - 1 \right) \right] \\ &= \frac{40\pi}{3} \text{ units}^3 \end{aligned}$$

2 a Volume = $\pi \int_1^3 \left(\frac{x^3}{x^2 + 1} \right)^2 dx$
 $\approx 5.926\pi$ {using technology}
 ≈ 18.6 units³

3 a $V = \pi \int_0^6 \left(\frac{x}{2} + 4 \right)^2 dx$
 $= \pi \int_0^6 \left(\frac{1}{4}x^2 + 4x + 16 \right) dx$
 $= \pi \left[\frac{x^3}{12} + \frac{4x^2}{2} + 16x \right]_0^6$
 $= \pi(18 + 72 + 96) - 0$
 $= 186\pi$ units³

c $V = \pi \int_0^4 (e^x)^2 dx$
 $= \pi \int_0^4 e^{2x} dx$
 $= \pi \left[\frac{1}{2}e^{2x} \right]_0^4$
 $= \pi \left(\frac{1}{2}e^8 - \frac{1}{2} \right)$
 $= \frac{\pi}{2}(e^8 - 1)$ units³

5 a a cone of base radius r and height h



b [AB] has gradient = $\frac{r-0}{0-h} = -\frac{r}{h}$
 \therefore its equation is $y = -\frac{r}{h}x + r$

6 a a sphere of radius r

b Volume = $\pi \int_0^2 (e^{\sin x})^2 dx$
 $\approx 9.613\pi$ {using technology}
 ≈ 30.2 units³

b $V = \pi \int_1^2 (x^2 + 3)^2 dx$
 $= \pi \int_1^2 (x^4 + 6x^2 + 9) dx$
 $= \pi \left[\frac{x^5}{5} + \frac{6x^3}{3} + 9x \right]_1^2$
 $= \pi \left[\left(\frac{32}{5} + 16 + 18 \right) - \left(\frac{1}{5} + 2 + 9 \right) \right]$
 $= \pi \left(\frac{146}{5} \right)$
 $= \frac{146\pi}{5}$ units³

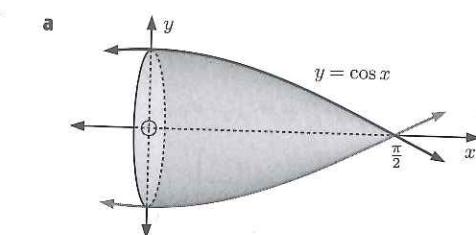
4 a Volume = $\pi \int_5^8 y^2 dx$
 $= \pi \int_5^8 (64 - x^2) dx$
 $= \pi \left[64x - \frac{x^3}{3} \right]_5^8$
 $= \pi \left[\left(512 - \frac{512}{3} \right) - \left(320 - \frac{125}{3} \right) \right]$
 $= 63\pi$ units³

b $63\pi \text{ cm}^3 \approx 198 \text{ cm}^3$

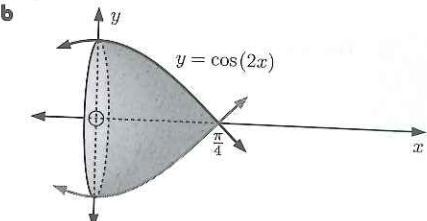
c $V = \pi \int_0^h \left(\frac{-r}{h}x + r \right)^2 dx$
 $= \pi r^2 \int_0^h \left(-\frac{x}{h} + 1 \right)^2 dx$
 $= \pi r^2 \int_0^h \left(\frac{x^2}{h^2} - \frac{2x}{h} + 1 \right) dx$
 $= \pi r^2 \left[\frac{x^3}{3h^2} - \frac{2x^2}{2h} + x \right]_0^h$
 $= \pi r^2 \left[\left(\frac{h}{3} - h + h \right) - 0 \right]$
 $= \frac{1}{3}\pi r^2 h$ units³

b $V = \pi \int_{-r}^r y^2 dx = 2\pi \int_0^r (r^2 - x^2) dx$

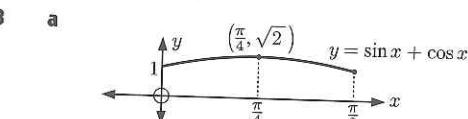
$$\begin{aligned} &= 2\pi \left[r^2x - \frac{x^3}{3} \right]_0^r \\ &= 2\pi(r^3 - \frac{r^3}{3} - 0) \\ &= 2\pi \times \frac{2}{3}r^3 \\ &= \frac{4}{3}\pi r^3 \text{ units}^3 \end{aligned}$$



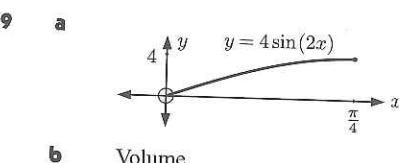
$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 dx \\ &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x dx \\ &= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cos(2x) \right) dx \\ &= \pi \left[\frac{1}{2}x + \frac{1}{2} \left(\frac{1}{2} \right) \sin(2x) \right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\frac{\pi}{4} + \frac{1}{4} \sin \pi - 0 \right] \\ &= \frac{\pi^2}{4} \text{ units}^3 \end{aligned}$$



$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} \cos^2(2x) dx \\ &= \pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\ &= \pi \left[\frac{1}{2}x + \frac{1}{2} \left(\frac{1}{4} \right) \sin(4x) \right]_0^{\frac{\pi}{4}} \\ &= \pi \left[\frac{\pi}{8} + \frac{1}{8} \sin \pi - 0 \right] \\ &= \frac{\pi^2}{8} \text{ units}^3 \end{aligned}$$



$$\begin{aligned} \text{b} \quad \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} (\sin x + \cos x)^2 dx \\ &= \pi \int_0^{\frac{\pi}{4}} (\sin^2 x + 2 \sin x \cos x + \cos^2 x) dx \\ &= \pi \int_0^{\frac{\pi}{4}} (1 + \sin(2x)) dx \\ &= \pi \left[x - \frac{1}{2} \cos(2x) \right]_0^{\frac{\pi}{4}} \\ &= \pi \left[\left(\frac{\pi}{4} - \frac{1}{2} \cos \left(\frac{\pi}{2} \right) \right) - (0 - \frac{1}{2} \cos 0) \right] \\ &= \pi \left(\frac{\pi}{4} + \frac{1}{2} \right) \text{ units}^3 \end{aligned}$$



$$\begin{aligned} \text{b} \quad \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} (4 \sin(2x))^2 dx \\ &= 16\pi \int_0^{\frac{\pi}{4}} \sin^2(2x) dx \\ &= 16\pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2} \cos(4x) \right) dx \\ &= 16\pi \left[\frac{x}{2} - \frac{1}{2} \left(\frac{1}{4} \right) \sin(4x) \right]_0^{\frac{\pi}{4}} \\ &= 16\pi \left[\left(\frac{\pi}{8} - \frac{1}{8} \sin \pi \right) - (0 - \frac{1}{8} \sin 0) \right] \\ &= 2\pi^2 \text{ units}^3 \end{aligned}$$

EXERCISE 22D.2

1 a The graphs meet where $4 - x^2 = 3$
 $\therefore x^2 = 1$
 $\therefore x = \pm 1$
 \therefore A is at $(-1, 3)$ and B is at $(1, 3)$.

$$\begin{aligned} \text{b} \quad V &= \pi \int_{-1}^1 ((4 - x^2)^2 - 3^2) dx \\ &= \pi \int_{-1}^1 (16 - 8x^2 + x^4 - 9) dx \\ &= \pi \int_{-1}^1 (x^4 - 8x^2 + 7) dx \\ &= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 7x \right]_{-1}^1 \\ &= \pi \left(\frac{1}{5} - \frac{8}{3} + 7 - \left(\frac{-1}{5} - \frac{-8}{3} - 7 \right) \right) \\ &= \frac{136\pi}{15} \text{ units}^3 \end{aligned}$$

2 a The graphs meet where $e^{\frac{x}{2}} = e$
 $\therefore e^{\frac{x}{2}} = e^1$
 $\therefore \frac{x}{2} = 1$
 $\therefore x = 2$
 \therefore A is at $(2, e)$.

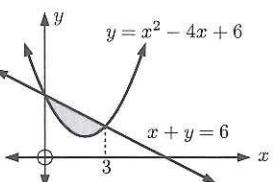
$$\begin{aligned} \text{b} \quad V &= \pi \int_0^2 \left(e^2 - (e^{\frac{x}{2}})^2 \right) dx \\ &= \pi \int_0^2 (e^2 - e^x) dx \\ &= \pi [e^2 x - e^x]_0^2 \\ &= \pi [(2e^2 - e^2) - (0 - 1)] \\ &= \pi(e^2 + 1) \text{ units}^3 \end{aligned}$$

- 3** **a** The graphs meet where $x = \frac{1}{x}$
 $\therefore x^2 = 1$
 $\therefore x = \pm 1$
 $\therefore x = 1$ {as $x > 0$ }
 $\therefore A$ is at $(1, 1)$.

b $V = \pi \int_1^2 \left(x^2 - \left(\frac{1}{x} \right)^2 \right) dx$
 $= \pi \int_1^2 (x^2 - x^{-2}) dx$
 $= \pi \left[\frac{x^3}{3} - \frac{x^{-1}}{-1} \right]_1^2$
 $= \pi \left[\left(\frac{8}{3} + \frac{1}{2} \right) - \left(\frac{1}{3} + 1 \right) \right]$
 $= \frac{11\pi}{6}$ units³

- 4** The graphs meet where $x^2 - 4x + 6 = 6 - x$
 $\therefore x^2 - 3x = 0$
 $\therefore x(x - 3) = 0$
 $\therefore x = 0$ or 3

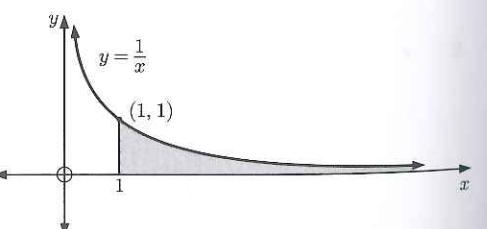
$\therefore V = \pi \int_0^3 [(6-x)^2 - (x^2 - 4x + 6)^2] dx$
 $= \pi \int_0^3 [(36 - 12x + x^2) - (x^4 - 4x^3 + 6x^2 - 4x^3 + 16x^2 - 24x + 6x^2 - 24x + 36)] dx$
 $= \pi \int_0^3 (-x^4 + 8x^3 - 27x^2 + 36x) dx$
 $= \pi \left[-\frac{x^5}{5} + 2x^4 - 9x^3 + 18x^2 \right]_0^3$
 $= \pi \left(-\frac{3^5}{5} + 2(3^4) - 9(27) + 18(9) - 0 \right)$
 $= \frac{162}{5}\pi$ units³



- 5** **a** The curves meet where $\sqrt{x-4} = 1$
 $\therefore x-4 = 1$
 $\therefore x = 5$
 $\therefore A$ is at $(5, 1)$.

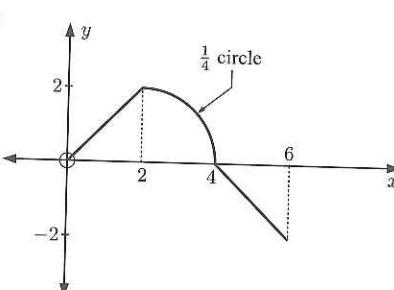
b $V = \pi \int_5^8 \left((\sqrt{x-4})^2 - 1^2 \right) dx$
 $= \pi \int_5^8 (x-4-1) dx$
 $= \pi \int_5^8 (x-5) dx$
 $= \pi \left[\frac{x^2}{2} - 5x \right]_5^8$
 $= \pi \left[(32-40) - \left(\frac{25}{2} - 25 \right) \right]$
 $= \frac{9\pi}{2}$ units³

- 6** The shaded area $= \int_1^\infty \frac{1}{x} dx$
 $= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx$
 $= \lim_{t \rightarrow \infty} [\ln(x)]_1^t, x > 0$
 $= \lim_{t \rightarrow \infty} \ln t$, which is infinite.



The volume of revolution $= \pi \int_1^\infty \left(\frac{1}{x} \right)^2 dx$
 $= \pi \lim_{t \rightarrow \infty} \int_1^t x^{-2} dx$
 $= \pi \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_1^t$
 $= \pi \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right)$
 $= \pi$, which is finite.

REVIEW SET 22A



a $\int_0^4 f(x) dx = \text{area of triangle} + \text{area of } \frac{1}{4} \text{ circle}$
 $= \frac{1}{2}(2 \times 2) + \frac{1}{4}\pi(2)^2$
 $= 2 + \pi$

b $\int_4^6 f(x) dx = -\text{area of triangle below } x\text{-axis}$
 $= -\frac{1}{2}(2 \times 2)$
 $= -2$

c $\int_0^6 f(x) dx = \int_0^4 f(x) dx + \int_4^6 f(x) dx$
 $= (2 + \pi) + (-2)$
 $= \pi$

2 shaded area $= \int_a^b [f(x) - g(x)] dx + \int_b^c [g(x) - f(x)] dx + \int_c^d [f(x) - g(x)] dx$

3 $\int_{-1}^3 f(x) dx$ gives us the correct area only if $f(x)$ is non-negative on the interval $-1 \leq x \leq 3$. In this case $f(x)$ is negative for $1 < x < 3$, so $\int_{-1}^3 f(x) dx$ does not provide the correct answer. (The shaded area which is below the x -axis is given by $\int_1^3 [0 - f(x)] dx = -\int_1^3 f(x) dx$.)

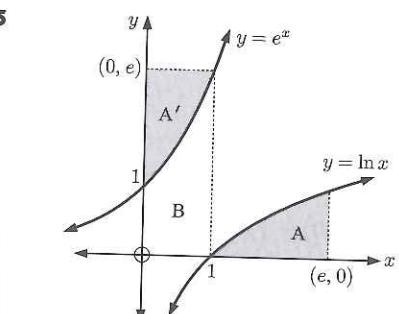
4 $y = k$ meets $y = x^2$ where $x^2 = k \therefore x = \pm\sqrt{k}$

By symmetry, $\int_0^{\sqrt{k}} (k - x^2) dx = \frac{1}{2} \times 5 \frac{1}{3} = \frac{1}{2} \times \frac{16}{3}$

$$\therefore \left[kx - \frac{x^3}{3} \right]_0^{\sqrt{k}} = \frac{8}{3}$$

$$\therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} = \frac{8}{3}$$

$$\therefore k = 4^{\frac{2}{3}} = \sqrt[3]{16}$$

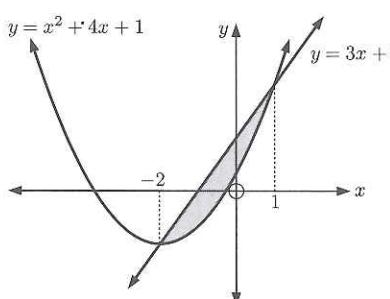


$y = e^x$ and $y = \ln x$ are inverse functions, so they are symmetrical about $y = x$
 \therefore area A = area A'

But area A' + area B = area of rectangle
 \therefore area A + area B = $e \times 1 = e$

Since area A $= \int_1^e \ln x dx$
and area B $= \int_0^1 e^x dx$,
 $\int_1^e \ln x dx + \int_0^1 e^x dx = e$

- 6 $y = x^2 + 4x + 1$ meets $y = 3x + 3$ where



$$x^2 + 4x + 1 = 3x + 3$$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x+2)(x-1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

$$\therefore \text{area} = \int_{-2}^1 [(3x+3) - (x^2+4x+1)] dx$$

$$= \int_{-2}^1 (-x^2 - x + 2) dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1$$

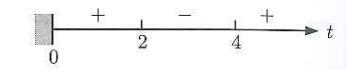
$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right)$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4$$

$$= 4\frac{1}{2} \text{ units}^2$$

- 7 a $v(t) = t^2 - 6t + 8 \text{ m s}^{-1}, t \geq 0$

$$= (t-4)(t-2) \quad \text{which has sign diagram:}$$



b Now $s(t) = \int (t^2 - 6t + 8) dt = \frac{t^3}{3} - 3t^2 + 8t + c$

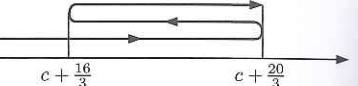
$$\therefore s(0) = c$$

$$s(2) = c + 6\frac{2}{3}$$

$$s(4) = c + 5\frac{1}{3}$$

$$s(5) = c + 6\frac{2}{3}$$

the motion diagram is:



The particle moves in the positive direction initially. When $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again when $t = 4$, $5\frac{1}{3}$ m from its starting point. When $t = 5$ it is $6\frac{2}{3}$ m from its starting point.

- c After 5 seconds, the particle is $6\frac{2}{3}$ m to the right of its starting point.

d The total distance travelled $= (c + \frac{20}{3} - c) + [(c + \frac{20}{3}) - (c + \frac{16}{3})] + [(c + \frac{20}{3}) - (c + \frac{16}{3})]$
 $= 9\frac{1}{3} \text{ m}$

- 8 Consider $y = 4e^x - 1$.

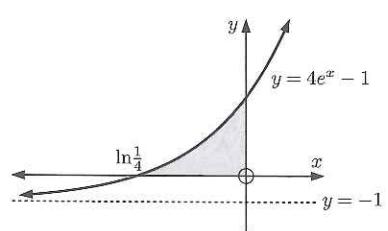
The x -intercept occurs when $y = 0$

$$\therefore 4e^x - 1 = 0$$

$$\therefore e^x = \frac{1}{4}$$

$$\therefore x = \ln \frac{1}{4} < 0$$

$y = 4e^x - 1$ is the graph of $y = e^x$ with a vertical stretch of factor 4 and a vertical translation of -1 .



$$\text{Area} = \int_{\ln \frac{1}{4}}^0 (4e^x - 1) dx$$

$$= [4e^x - x]_{\ln \frac{1}{4}}^0$$

$$= (4e^0 - 0) - \left(4e^{\ln \frac{1}{4}} - \ln \frac{1}{4} \right)$$

$$= 4 - 0 - 4(\frac{1}{4}) + \ln \frac{1}{4}$$

$$= 3 + \ln \frac{1}{4} \text{ units}^2 \quad (= 3 - \ln 4 \text{ units}^2)$$

- 9 $y = x^2$ and $y = 4$ meet where $x^2 = 4$

$$\therefore x = \pm 2$$

But $x > 0$, so $x = 2$

$$\begin{aligned} \text{Hence } V &= \pi \int_0^2 (4^2 - (x^2)^2) dx \\ &= \pi \int_0^2 (16 - x^4) dx \\ &= \pi \left[16x - \frac{x^5}{5} \right]_0 \\ &= \pi \left(32 - \frac{32}{5} \right) \\ &= \frac{128\pi}{5} \text{ units}^3 \end{aligned}$$

REVIEW SET 22B

- 1 $v(t) = 2t - 3t^2 = t(2 - 3t)$ which has sign diagram:

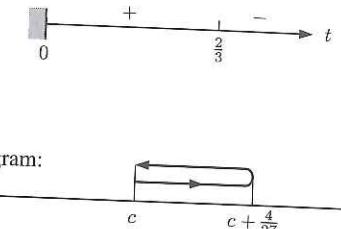
$$\text{Now } s(t) = \int (2t - 3t^2) dt = t^2 - t^3 + c \text{ metres}$$

and so $s(0) = c$

$$s(\frac{2}{3}) = \frac{4}{9} - \frac{8}{27} + c = c + \frac{4}{27}$$

$$s(1) = 1 - 1 + c = c$$

with motion diagram:



$$\therefore \text{total distance travelled} = (c + \frac{4}{27} - c) + (c + \frac{4}{27} - c) = \frac{8}{27} \text{ m} \approx 29.6 \text{ cm}$$

- 2 a $f(x) = \frac{x}{1+x^2} \quad \therefore f'(x) = \frac{1(1+x^2) - x(2x)}{(1+x^2)^2}$ {quotient rule}

$$= \frac{1+x^2 - 2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{(1+x)(1-x)}{(1+x^2)^2}$$

which has sign diagram:



∴ there is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(1, \frac{1}{2})$.

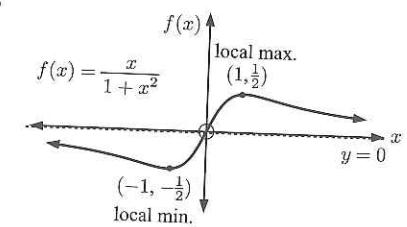
- b As $x \rightarrow \infty$, $f(x) \rightarrow 0$ (above).

As $x \rightarrow -\infty$, $f(x) \rightarrow 0$ (below).

d Area $= \int_{-2}^0 \left[0 - \frac{x}{1+x^2} \right] dx$

$$= \int_{-2}^0 \frac{-x}{1+x^2} dx$$

$$\approx 0.805 \text{ units}^2$$



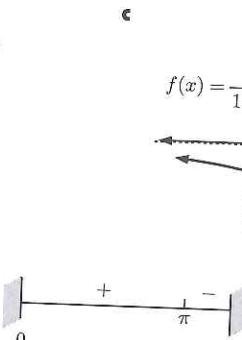
- 3 $v(t) = \sin t$ which has sign diagram:

$$\text{Now } s(t) = \int \sin t dt = -\cos t + c \text{ metres}$$

$$\therefore s(0) = -1 + c$$

$$s(\pi) = 1 + c$$

$$s(4) = -\cos 4 + c \approx c + 0.654$$



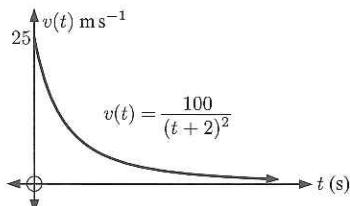
$$\therefore \text{total distance travelled} = [(c+1) - (c-1)] + [(c+1) - (c+0.654)] \approx 2.35 \text{ m}$$

4 $v(t) = \frac{100}{(t+2)^2} = 100(t+2)^{-2} \text{ ms}^{-1}$

a At $t=0$, $v(0) = \frac{100}{2^2} = 25 \text{ ms}^{-1}$. At $t=3$, $v(3) = \frac{100}{5^2} = 4 \text{ ms}^{-1}$.

b As $t \rightarrow \infty$, $v(t) \rightarrow 0 \text{ ms}^{-1}$ (above)

c



d As $v(t)$ is always positive, the boat is

always travelling forwards.

$$\begin{aligned}s(t) &= \int v(t) dt \\&= \int 100(t+2)^{-2} dt \\&= -100(t+2)^{-1} + c \\&= \frac{-100}{t+2} + c\end{aligned}$$

$\therefore s(0) = c - 50 \text{ m}$

when the boat has travelled 30 m,

$$s(t) = c - 20$$

$$\therefore c - 20 = \frac{-100}{t+2} + c$$

$$\therefore \frac{-100}{t+2} = -20$$

$$\therefore t+2 = 5$$

$$\therefore t = 3 \text{ seconds}$$

$$\begin{aligned}\text{f } \frac{dv}{dt} &= \frac{-200}{(t+2)^3} = -\frac{1}{5} \frac{1000}{(t+2)^3} \\&= -\frac{1}{5} \left(\frac{100}{(t+2)^2} \right)^{\frac{3}{2}} \\&= -\frac{1}{5} v^{\frac{3}{2}} \\&\therefore \frac{dv}{dt} = -kv^{\frac{3}{2}} \text{ where } k = \frac{1}{5}\end{aligned}$$

5 a The graphs meet when $\cos 2x = e^{3x}$

Using technology, $x = 0$ and $x \approx -0.7292$

b Shaded area $\approx \int_{-0.7292}^0 (\cos 2x - e^{3x}) dx \approx 0.2009 \text{ units}^2$ {using technology}

6 $C'(x) = 2 + 8e^{-x}$

$$\therefore C(x) = \int (2 + 8e^{-x}) dx = 2x - 8e^{-x} + c$$

But $C(0) = 240$, so $2(0) - 8e^0 + c = 240$
 $\therefore c = 248$

So $C(x) = 2x - 8e^{-x} + 248$

$\therefore C(80) = 2(80) - 8e^{-80} + 248 \approx 408$

\therefore the total cost is £408 per day.

8 a The graphs meet where

$$x^2 = \sin x$$

$\therefore x = 0$ or ≈ 0.8767 {using technology}

$\therefore a \approx 0.8767$

7 $\int_0^m \sin x dx = \frac{1}{2}$

$$\therefore [-\cos x]_0^m = \frac{1}{2}$$

$$\therefore -\cos m + \cos 0 = \frac{1}{2}$$

$$\therefore \cos m = \frac{1}{2}$$

$$\therefore m = \frac{\pi}{3} \quad \{0 < m < \frac{\pi}{2}\}$$

b area $\approx \int_0^{0.8767} (\sin x - x^2) dx$

$\approx 0.1357 \text{ units}^2$ {using technology}

9 a $y = \cos(2x)$ meets the x -axis where $2x = \frac{\pi}{2}$, or $x = \frac{\pi}{4}$.

$$\therefore V = \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2(2x) dx = \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx$$

$$\begin{aligned}&= \pi \left[\frac{1}{2}x + \frac{1}{8} \sin(4x) \right]_{\frac{\pi}{16}}^{\frac{\pi}{4}} \\&= \pi \left[\left(\frac{\pi}{8} + \frac{1}{8} \sin(\pi) \right) - \left(\frac{\pi}{32} + \frac{1}{8} \sin\left(\frac{\pi}{4}\right) \right) \right] \\&= \pi \left(\frac{\pi}{8} - \frac{\pi}{32} - \frac{1}{8} \left(\frac{1}{\sqrt{2}} \right) \right) \\&= \pi \left(\frac{3\pi}{32} - \frac{1}{8\sqrt{2}} \right) \text{ units}^3\end{aligned}$$

b $V = \pi \int_0^2 (e^{-x} + 4)^2 dx$

$$= \pi \int_0^2 (e^{-2x} + 8e^{-x} + 16) dx$$

$$= \pi \left[-\frac{1}{2}e^{-2x} + \frac{8}{-1}e^{-x} + 16x \right]_0^2$$

$$= \pi \left[\left(-\frac{1}{2}e^{-4} - 8e^{-2} + 32 \right) - \left(-\frac{1}{2} - 8 \right) \right]$$

$$= \pi \left(\frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2} \right) \text{ units}^3$$

$$\approx 124 \text{ units}^3$$

REVIEW SET 22C

1 $a(t) = 6t - 30 \text{ cms}^{-2}$

$$\therefore v(t) = \int (6t - 30) dt = 3t^2 - 30t + c$$

But $v(0) = 27$ so $c = 27$

$$\therefore v(t) = 3t^2 - 30t + 27 \text{ cm s}^{-1}$$

$$\therefore s(t) = \int (3t^2 - 30t + 27) dt = t^3 - 15t^2 + 27t + d$$

But $s(0) = 0$ so $d = 0$

$$\therefore s(t) = t^3 - 15t^2 + 27t$$

Also, $v(t) = 3t^2 - 30t + 27$

$$= 3(t^2 - 10t + 9)$$

$$= 3(t-1)(t-9)$$

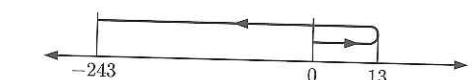
The particle comes to rest for the second time at $t = 9$ seconds.

$$s(0) = 0 \text{ cm}$$

$$s(1) = 1^3 - 15(1)^2 + 27(1) = 13 \text{ cm}$$

$$s(9) = 9^3 - 15(9)^2 + 27(9) = -243 \text{ cm}$$

Motion diagram:

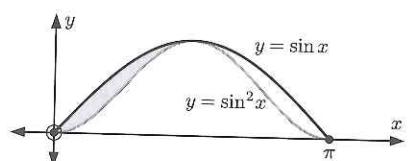


the total distance travelled is $13 + (13 - (-243)) = 269 \text{ cm}$

which has sign diagram:



2 a



b Area $= \int_0^{\frac{\pi}{2}} (\sin x - \sin^2 x) dx$

$$= \int_0^{\frac{\pi}{2}} (\sin x - (\frac{1}{2} - \frac{1}{2} \cos 2x)) dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin x + \frac{1}{2} \cos 2x - \frac{1}{2}) dx$$

$$= \left[-\cos x + \frac{1}{4} \sin 2x - \frac{1}{2}x \right]_0^{\frac{\pi}{2}}$$

$$= \left(0 + \frac{1}{4}(0) - \frac{\pi}{4} \right) - \left(-1 + 0 - 0 \right)$$

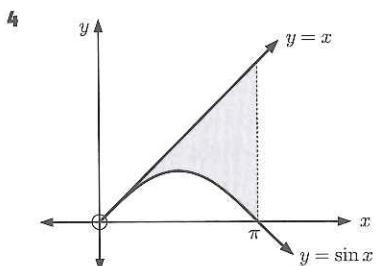
$$= \left(1 - \frac{\pi}{4} \right) \text{ units}^2$$

- 3 The area between $x = 0$ and $x = a$ is 2 units².

$$\begin{aligned}\therefore \int_0^a e^x dx &= 2 \\ \therefore [e^x]_0^a &= 2 \\ \therefore e^a - e^0 &= 2 \\ \therefore e^a &= 3 \\ \therefore a &= \ln 3\end{aligned}$$

The area between $x = a = \ln 3$ and $x = b$ is 2 units².

$$\begin{aligned}\therefore \int_{\ln 3}^b e^x dx &= 2 \\ \therefore [e^x]_{\ln 3}^b &= 2 \\ \therefore e^b - e^{\ln 3} &= 2 \\ \therefore e^b - 3 &= 2 \\ \therefore e^b &= 5 \\ \therefore b &= \ln 5\end{aligned}$$

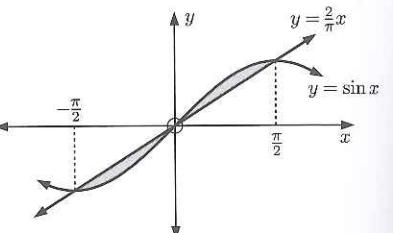


Required area = area of Δ – area under sine curve

$$\begin{aligned}&= \frac{1}{2}\pi \times \pi - \int_0^\pi \sin x dx \\ &= \frac{\pi^2}{2} - [-\cos x]_0^\pi \\ &= \frac{\pi^2}{2} - [-\cos \pi + \cos 0] \\ &= \left(\frac{\pi^2}{2} - 2\right) \text{ units}^2\end{aligned}$$

- 5 The graphs meet when $\frac{2}{\pi}x = \sin x$

$$\begin{aligned}\therefore x &= -\frac{\pi}{2}, 0, \frac{\pi}{2} \quad \{\text{using technology}\} \\ \therefore \text{area} &= \int_{-\frac{\pi}{2}}^0 \left(\frac{2}{\pi}x - \sin x\right) dx + \int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2}{\pi}x\right) dx \\ &= \left[\frac{x^2}{\pi} + \cos x\right]_{-\frac{\pi}{2}}^0 + \left[-\cos x - \frac{x^2}{\pi}\right]_0^{\frac{\pi}{2}} \\ &= (0+1) - (\frac{\pi}{4}+0) + (0-\frac{\pi}{4}) - (-1-0) \\ &= (2 - \frac{\pi}{2}) \text{ units}^2\end{aligned}$$



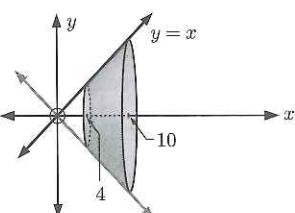
- 6 The coordinates of B are $(2, 4+k)$

$$\begin{aligned}\therefore \text{area rectangle OABC} &= 2 \times (4+k) \\ &= 8+2k \\ \therefore \text{since the two shaded regions are equal in area, each area is } &4+k \text{ units}^2.\end{aligned}$$

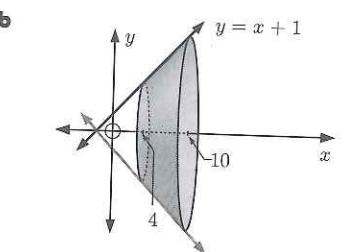
$$\therefore \int_0^2 (x^2 + k) dx = 4+k$$

$$\begin{aligned}\therefore \left[\frac{x^3}{3} + kx\right]_0^2 &= 4+k \\ \therefore \frac{8}{3} + 2k &= 4+k \\ \therefore k &= 4 - \frac{8}{3} \\ \therefore k &= \frac{4}{3}\end{aligned}$$

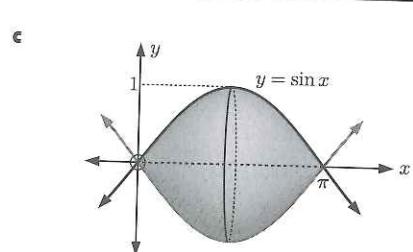
- 7 a



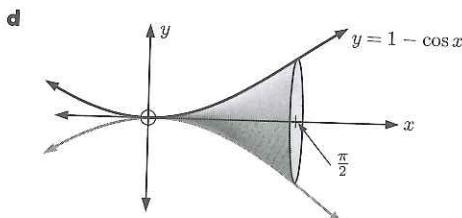
$$\begin{aligned}V &= \pi \int_4^{10} x^2 dx \\ &= \pi \left[\frac{x^3}{3}\right]_4^{10} \\ &= \pi \left(\frac{1000}{3} - \frac{64}{3}\right) \\ &= \frac{936\pi}{3} \\ &= 312\pi \text{ units}^3\end{aligned}$$



$$\begin{aligned}V &= \pi \int_4^{10} (x+1)^2 dx \\ &= \pi \left[\frac{(x+1)^3}{3}\right]_4^{10} \\ &= \pi \left(\frac{11^3}{3} - \frac{5^3}{3}\right) \\ &= \frac{1206\pi}{3} = 402\pi \text{ units}^3\end{aligned}$$



$$\begin{aligned}V &= \pi \int_0^\pi \sin^2 x dx \\ &= \pi \int_0^\pi \left(\frac{1}{2} - \frac{1}{2} \cos(2x)\right) dx \\ &= \pi \left[\frac{1}{2}x - \frac{1}{2} \left(\frac{1}{2}\right) \sin(2x)\right]_0^\pi \\ &= \pi \left(\frac{1}{2}\pi - \frac{1}{4} \sin 2\pi\right) \\ &= \frac{\pi^2}{2} \text{ units}^3\end{aligned}$$



$$\begin{aligned}V &= \pi \int_0^{\frac{\pi}{2}} (1 - \cos x)^2 dx \\ &= \pi \int_0^{\frac{\pi}{2}} (1 - 2\cos x + \cos^2 x) dx \\ &= \pi \int_0^{\frac{\pi}{2}} (1 - 2\cos x + \frac{1}{2} + \frac{1}{2} \cos 2x) dx \\ &= \pi \left[\frac{3}{2}x - 2\sin x + \frac{1}{4} \sin 2x\right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\left(\frac{3}{2} \cdot \frac{\pi}{2}\right) - 2\sin\left(\frac{\pi}{2}\right) + \frac{1}{4} \sin \pi\right] - \left[\frac{3}{2}(0) - 2\sin 0 + \frac{1}{4} \sin 0\right] \\ &= \pi \left(\frac{3\pi}{4} - 2\right) \text{ units}^3\end{aligned}$$

- 8 $y = \sin x$ and $y = \cos x$ meet where $\sin x = \cos x$

$$\begin{aligned}\therefore \frac{\sin x}{\cos x} &= 1 \\ \therefore \tan x &= 1 \\ \therefore x &= \frac{\pi}{4}\end{aligned}$$

$$\text{Hence } V = \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) dx$$

$$\begin{aligned}&= \pi \int_0^{\frac{\pi}{4}} \cos(2x) dx \\ &= \pi \left[\frac{1}{2} \sin(2x)\right]_0^{\frac{\pi}{4}} \\ &= \pi \left(\frac{1}{2} \sin\left(\frac{\pi}{2}\right) - \frac{1}{2} \sin 0\right) \\ &= \pi \left(\frac{1}{2}(1) - 0\right) \\ &= \frac{\pi}{2} \text{ units}^3\end{aligned}$$

$$9 \quad \frac{dT}{dx} = \frac{k}{x} = kx^{-1}$$

$$\therefore T = k \ln x + c \quad \{x > 0\}$$

When $x = r_1$, $T = T_0$

$$\therefore k \ln r_1 + c = T_0$$

$$\therefore T = k \ln x + T_0 - k \ln r_1$$

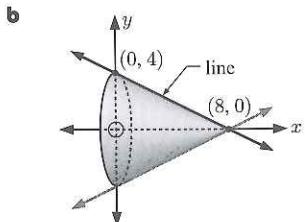
$$= T_0 + k \ln\left(\frac{x}{r_1}\right)$$

So, when $x = r_2$,

$$T = T_0 + k \ln\left(\frac{r_2}{r_1}\right)$$

\therefore the outer surface has temperature $T_0 + k \ln\left(\frac{r_2}{r_1}\right)$

10 a $V = \frac{1}{3}\pi r^2 h$
 $= \frac{1}{3}\pi \times 4^2 \times 8$
 $= \frac{1}{3}\pi \times 128$
 $= \frac{128\pi}{3}$ units³



gradient $= \frac{0-4}{8-0} = -\frac{1}{2}$
 \therefore the line has equation $y = -\frac{1}{2}x + 4$
 $\therefore V = \pi \int_0^8 \left(-\frac{1}{2}x + 4\right)^2 dx$
 $= \pi \int_0^8 \left(\frac{x^2}{4} - 4x + 16\right) dx$
 $= \pi \left[\frac{x^3}{12} - \frac{4x^2}{2} + 16x\right]_0^8$
 $= \pi \left(\frac{128}{3} - 128 + 128 - 0\right)$
 $= \frac{128\pi}{3}$ units³ ✓

Chapter 23

STATISTICAL DISTRIBUTIONS OF DISCRETE RANDOM VARIABLES

EXERCISE 23A

- 1** a The quantity of fat in a sausage is a continuous random variable.
 b The mark out of 50 for a geography test is a discrete random variable.
 c The weight of a seventeen year old student is a continuous random variable.
 d The volume of water in a cup of coffee is a continuous random variable.
 e The number of trout in a lake is a discrete random variable.
 f The number of hairs on a cat is a discrete random variable.
 g The length of hairs on a horse is a continuous random variable.
 h The height of a sky-scraper is a continuous random variable.
- 2** a i The random variable X is the height of water in the rain gauge.
 ii $0 \leq X \leq 200$ mm iii The variable is a continuous random variable.
 b i The random variable X is the stopping distance.
 ii $0 \leq X \leq 50$ m iii The variable is a continuous random variable.
 c i The random variable X is the number of times that the switch is turned on or off before it fails.
 ii X any integer ≥ 1 iii The variable is a discrete random variable.

- 3** a Since X is the number of weighing devices that are accurate, $X = 0, 1, 2, 3$ or 4.

	YYNN	YNYN	NNNY
	YYYN	YNNY	NNYN
	YYNY	NYYN	NNYN
	YNYY	NYNY	NYNN
($X = 4$)	($X = 3$)	($X = 2$)	($X = 1$) ($X = 0$)

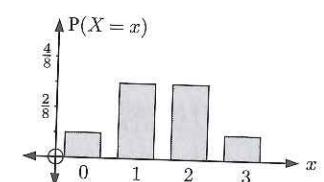
- c i If two are accurate then $X = 2$.
 ii If at least two are accurate then 2, 3 or 4 are accurate $\therefore X = 2, 3$ or 4.

- 4** a If 3 coins are tossed then the number of heads X can be 0, 1, 2 or 3.

b Suppose H represents heads, T represents tails.	$P(X = 0) = \frac{1}{8}$	$P(X = 1) = \frac{3}{8}$
HHT	TTH	
HTH	THT	$P(X = 2) = \frac{3}{8}$

HHH	THH	HTT	TTT
($X = 3$)	($X = 2$)	($X = 1$)	($X = 0$)

d



EXERCISE 23B

1 a $\sum_{x=0}^2 P(x) = 1$
 $\therefore 0.3 + k + 0.5 = 1$
 $\therefore k = 0.2$

b $\sum_{x=0}^3 P(X = x) = 1$
 $\therefore k + 2k + 3k + k = 1$
 $\therefore 7k = 1$
 $\therefore k = \frac{1}{7}$