Inverse & Composite Functions

Chapter 23 Supplement

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10-3 Composition and Inverses of Functions

Objective To find the composite of two given functions and to find the inverse of a given function.

Consider the squaring function $f(x) = x^2$ and the doubling function $g(x) = 2x$. As the diagram below shows, these two functions can be combined to produce a new function whose value at x is $f(g(x))$, read "f of g of x."

$$\begin{array}{lcc}
 & g \text{ is the doubling function} & f \text{ is the squaring function} \\
3 \longrightarrow & g(3) = 6 \longrightarrow & f(g(3)) = f(6) = 36 \\
5 \longrightarrow & g(5) = 10 \longrightarrow & f(g(5)) = f(10) = 100 \\
x \longrightarrow & g(x) = 2x \longrightarrow & f(g(x)) = f(2x) = (2x)^2
\end{array}$$

Notice that $f(g(x))$ is evaluated by working from the innermost parentheses to the outside. You begin with x, then g doubles x, and then f squares the result.

$$f(g(x)) = f(2x) = (2x)^2$$

The function whose value at x is $f(g(x))$ is called the **composite** of the functions f and g. The operation that combines f and g to produce their composite is called **composition.**

Example 1 If $f(x) = 3x - 5$ and $g(x) = \sqrt{x}$, find the following.

a. $f(g(4))$ **b.** $g(f(4))$
c. $f(g(x))$ **d.** $g(f(x))$

Solution
a. Since $g(4) = \sqrt{4} = 2$, $f(g(4)) = 3 \cdot 2 - 5 = 1$.
b. Since $f(4) = 3 \cdot 4 - 5 = 7$, $g(f(4)) = \sqrt{7}$.
c. $f(g(x)) = f(\sqrt{x}) = 3\sqrt{x} - 5$
d. $g(f(x)) = g(3x - 5) = \sqrt{3x - 5}$

Notice in Example 1 that $f(g(x)) \neq g(f(x))$.

The function $I(x) = x$ is called the **identity function.** It behaves like the multiplicative identity 1.

$$\begin{array}{ll} a \cdot 1 = a & \text{for all numbers } a \\ f(I(x)) = f(x) & \text{for all functions } f \end{array}$$

For the two functions f and g defined in the example at the top of the next page, the composites $f(g(x))$ and $g(f(x))$ are both equal to the identity function.

Example 2 If $f(x) = \frac{x+4}{2}$ and $g(x) = 2x - 4$, find the following.

a. $g(1)$ and $f(g(1))$ **b.** $f(-3)$ and $g(f(-3))$ **c.** $f(g(x))$ **d.** $g(f(x))$

Solution **a.** $g(1) = -2$ and $f(g(1)) = f(-2) = 1$. *Answer*

Notice that g: $1 \to -2$ and f: $-2 \to 1$.

b. $f(-3) = \frac{1}{2}$ and $g(f(-3)) = g\left(\frac{1}{2}\right) = -3$. *Answer*

Notice that f: $-3 \to \frac{1}{2}$ and g: $\frac{1}{2} \to -3$.

Parts (a) and (b) suggest that the functions f and g "undo each other." Parts (c) and (d) prove that this is so for any number x.

c. $f(g(x)) = f(2x - 4) = \frac{2x - 4 + 4}{2} = x$ *Answer*

d. $g(f(x)) = g\left(\frac{x+4}{2}\right) = 2\left(\frac{x+4}{2}\right) - 4 = x$ *Answer*

In multiplication, two numbers whose product is the identity 1, such as 2 and 2^{-1}, are called inverses. Similarly, two functions whose composite is the identity I, such as f and g in Example 2, are called *inverse functions*.

Inverse Functions

The functions f and g are **inverse functions if**

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g$$

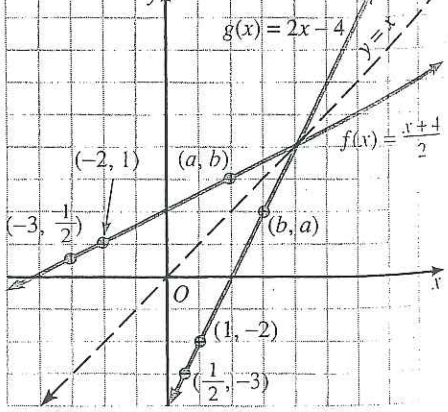
and

$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

The inverse of a function f is usually denoted f^{-1}, read "f inverse."

Caution: The superscript -1 in f^{-1} is *not* an exponent. The symbol $f^{-1}(x)$ denotes the value of f inverse at x; it does *not* mean $\frac{1}{f(x)}$.

Suppose that two functions f and g are inverses and that $f(a) = b$, so that (a, b) is on the graph of f. Then $g(b) = g(f(a)) = a$, and the point (b, a) must be on the graph of g. This means every point (a, b) on the graph of f corresponds to a point (b, a) on the graph of g. Therefore, the graphs are *mirror images* of each other with respect to the line $y = x$. You can verify this by drawing graphs of inverse functions on the same axes. A computer or a graphing calculator may be helpful. The diagram shows the inverse functions of Example 2.



Some functions do not have inverse functions. If the reflection of the graph of a function f is itself to be the graph of a function, the graph of f must *not* contain two different points with the same y-coordinate. Therefore, a function has an inverse function if and only if it is one-to-one.

As you learned in Lesson 10-2, every horizontal line intersects the graph of a one-to-one function in at most one point. Therefore, you can use the *horizontal-line test* to tell whether a given function has an inverse function.

Horizontal-Line Test

A function has an inverse function if and only if every horizontal line intersects the graph of the function in *at most* one point.

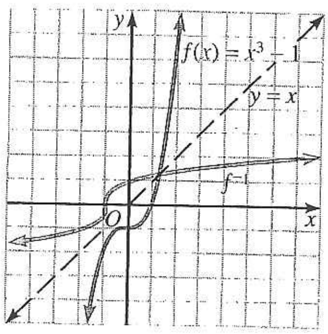
If a function has an inverse function, you can find it by writing y for $f(x)$, interchanging x and y, and solving for y. Example 3 illustrates.

Example 3 Let $f(x) = x^3 - 1$.

a. Graph f and determine whether f has an inverse function. If so, graph f^{-1} by reflecting f across the line $y = x$.

b. Find $f^{-1}(x)$.

Solution **a.** The graph of f is shown in red at the right. The graph of f passes the horizontal-line test, so f has an inverse. The graph of f^{-1}, shown in blue, is the reflection of the graph of f across the line $y = x$.



b. Replace $f(x)$ by y: $\quad y = x^3 - 1$

Interchange x and y: $\quad x = y^3 - 1$

Solve for y: $\quad y^3 = x + 1$

$$y = \sqrt[3]{x+1}$$

$$f^{-1}(x) = \sqrt[3]{x+1} \quad \textit{Answer}$$

Oral Exercises

Suppose $f(x) = 3x$, $g(x) = x + 1$, and $h(x) = x^2 + 2$. Find the following.

1. a. $f(g(3))$ b. $f(g(0))$ c. $f(g(-6))$ d. $f(g(x))$
2. a. $g(f(4))$ b. $g(f(5))$ c. $g(f(-6))$ d. $g(f(x))$
3. a. $f(h(2))$ b. $h(f(2))$ c. $f(h(x))$ d. $h(f(x))$
4. a. $g(h(3))$ b. $h(g(3))$ c. $g(h(x))$ d. $h(g(x))$
5. Find $f^{-1}(x)$.
6. Find $g^{-1}(x)$.
7. Does h^{-1} exist? Why or why not?

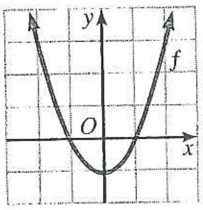
Written Exercises

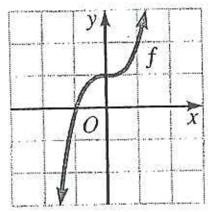
Suppose $f(x) = \frac{x}{2}$, $g(x) = x - 3$, and $h(x) = \sqrt{x}$. Find a real-number value or an expression in x for each of the following. If no real value can be found, say so.

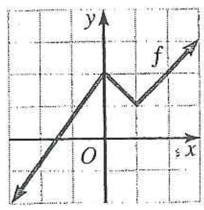
A

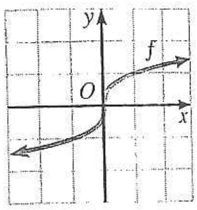
1. a. $f(g(8))$ b. $f(g(-5))$ c. $f(g(0))$ d. $f(g(x))$
2. a. $g(f(8))$ b. $g(f(-5))$ c. $g(f(0))$ d. $g(f(x))$
3. a. $f(h(9))$ b. $f(h(4))$ c. $f(h(-4))$ d. $f(h(x))$
4. a. $h(f(32))$ b. $h(f(16))$ c. $h(f(x))$ d. $f(f(x))$
5. a. $h(g(12))$ b. $h(g(2))$ c. $h(g(x))$ d. $h(h(x))$
6. a. $g(h(9))$ b. $g(h(\sqrt{3}))$ c. $g(h(x))$ d. $g(g(x))$

Use the horizontal-line test to determine whether each function f has an inverse function. If so, draw a rough sketch of f^{-1} by reflecting f across $y = x$.

7. 

8. 

9. 

10. 

In Exercises 11–14, find $f^{-1}(x)$. Then graph f and f^{-1} in the same coordinate system. You may wish to verify your graphs on a computer or a graphing calculator.

11. $f(x) = 2x - 3$ 12. $f(x) = \frac{x + 6}{3}$ 13. $f(x) = x^3$ 14. $f(x) = \frac{12}{x}$

In Exercises 15–22, graph g and use the horizontal-line test to determine if g has an inverse function. If so, find $g^{-1}(x)$. If g has no inverse, say so. You may wish to verify your graphs on a computer or a graphing calculator.

15. $g(x) = \left(\frac{8}{x}\right)^3$ 16. $g(x) = \sqrt[3]{2x}$ 17. $g(x) = x^4$ 18. $g(x) = |x|$

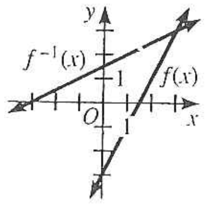
B

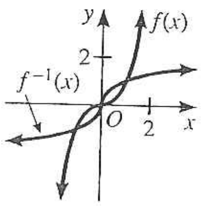
19. $g(x) = x^2 - x$ 20. $g(x) = x^3 + 2$ 21. $g(x) = \sqrt{x^2}$ 22. $g(x) = (2x + 3)^5$

23. a. Draw the graph of $f(x) = 2^x$ by making a table of values and carefully plotting several points.
    b. Draw the graph of f^{-1} on the same coordinate system by reflecting the graph of f in the line $y = x$.
    c. Find $f^{-1}(2)$, $f^{-1}(4)$, $f^{-1}(8)$, and $f^{-1}(\frac{1}{2})$.
    d. Give the domain and range of f and f^{-1}.

Written Exercises, pages 466–467 **1. a.** $\frac{5}{2}$ **b.** -4 **c.** $-\frac{3}{2}$ **d.** $\frac{x-3}{2}$ **3. a.** $\frac{3}{2}$ **b.** 1 **c.** not real **d.** $\frac{\sqrt{x}}{2}$ **5. a.** 3 **b.** not real **c.** $\sqrt{x-3}$ **d.** $\sqrt[4]{x}$ **7.** no **9.** no

11. $f^{-1}(x) = \frac{x+3}{2}$ **13.** $f^{-1}(x) = \sqrt[3]{x}$





15. $g^{-1}(x) = \frac{8}{\sqrt[3]{x}}$ **17.** no inverse **19.** no inverse **21.** no inverse **23. c.** 1; 2; 3; -1 **d.** f: $D = \{\text{reals}\}$, $R = \{y: y > 0\}$; f^{-1}: $D = \{x: x > 0\}$, $R = \{\text{reals}\}$ **25.** $m = 1$, $b = 0$; or $m = -1$

Mixed Review Exercises, page 467 **1.** $\frac{1}{125}$ **2.** $7\sqrt{2}$ **3.** $-5 - 12i$ **4.** 81 **5.** -24 **6.** $\frac{1}{16}$ **7.** 9 **8.** $5 - 4\sqrt{2}$ **9.** -3 **10.** 128 **11.** $1 + 2i$ **12.** 2

Self-Test 1, page 467 **1. a.** $2x^{2/3}y^{-1/3}$ **b.** $6^{-1/3}$ **2. a.** $\frac{25\sqrt{5}}{2}$ **b.** $x^2y\sqrt[6]{xy^5}$ **3.** $\{34\}$ **4. a.** $2^{-5\sqrt{2}}$ **b.** $2^{-8\sqrt{5}}$ **5.** $\{3\}$ **6. a.** 13 **b.** 5 **c.** $6\sqrt{x} + 1$ **d.** $\sqrt{6x+1}$

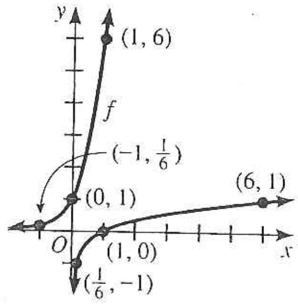
7. $f(g(x)) = 3\left(\frac{x+7}{3}\right) - 7 = x + 7 - 7 = x$;

$g(f(x)) = \frac{(3x-7)+7}{3} = \frac{3x}{3} = x$

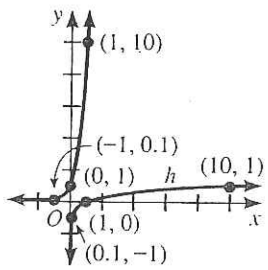
Written Exercises, pages 470–472 **1.** 3 **3.** 4 **5.** 0 **7.** -2 **9.** $\frac{3}{2}$ **11.** $\frac{1}{4}$ **13.** $\frac{2}{3}$ **15.** -3 **17.** $-\frac{2}{3}$ **19.** $\{49\}$ **21.** $\left\{\frac{1}{3}\right\}$ **23.** $\left\{\frac{1}{8}\right\}$ **25.** $\{9\}$ **27.** $\left\{\frac{1}{49}\right\}$ **29.** $\{x: x > 0 \text{ and } x \neq 1\}$

31. a. $3 + 2 = 5$ **b.** $\frac{1}{2} + \frac{3}{2} = 2$ **c.** $\log_b M + \log_b N = \log_b MN$ **33. a.** $\log_6 x$ **b.** 2; $-\frac{1}{2}$ **c.** f: $D = \{\text{reals}\}$, $R = \{y: y > 0\}$; f^{-1}: $D = \{x: x > 0\}$, $R = \{\text{reals}\}$

35.



37.



39. $\{3\}$
41. positive
43. a. 120 dB
b. 10^4

Written Exercises, pages 476–477

1. $6 \log_2 M + 3 \log_2 N$ **3.** $\log_2 M + \frac{1}{2} \log_2 N$ **5.** $4 \log_2 M - 3 \log_2 N$ **7.** $\frac{1}{2} \log_2 M - \frac{3}{2} \log_2 N$ **9.** 1.90 **11.** 0.15 **13.** 0.90 **15.** 0.35 **17.** -0.95 **19.** -0.22 **21.** $\log_4 p^5q$ **23.** $\log_3 \frac{A^4}{\sqrt{B}}$ **25.** $\log_2 8MN$ **27.** $\log_5 \frac{5}{x^3}$ **29.** 2 **31.** $\frac{3}{2}$ **33.** $\{45\}$ **35.** $\{1\}$ **37.** $\{6\}$ **39.** $\{\pm 5\}$ **41. a.** 6 **b.** $\frac{1}{4}$ **c.** 1 **45.** $\{3\}$ **47.** $\{2\}$ **49.** $\{\sqrt{85}\}$ **51.** $\{2\}$

Mixed Review Exercises, page 477 **1.** $\left\{\frac{\sqrt{2}}{2}\right\}$ **2.** $\{-3, 1, 2\}$ **3.** $\{5\}$ **4.** $\{4\}$ **5.** $\left\{-\frac{3}{2}\right\}$ **6.** $\{7\}$ **7.** $\left\{-\frac{1}{2}\right\}$ **8.** $\{4\}$ **9.** $\{2 \pm \sqrt{5}\}$ **10.** 3 **11.** 2 **12.** 1 **13.** 1

Self-Test 2, page 477 **1. a.** $3^4 = 81$ **b.** $6^3 = 216$ **2. a.** $\log_5 625 = 4$ **b.** $\log_{25} 125 = \frac{3}{2}$ **3. a.** 3 **b.** 12 **4.** $\{3\}$ **5.** $\frac{5}{3} \log_2 M + 2 \log_2 N$ **6.** -1.40 **7.** $\{3\}$

Written Exercises, pages 481–482 **1.** 1.79 **3.** 0.00792 **5.** 575 **7.** 33.7 **9.** 7.13 **11.** 692 **13.** 0.0158 **15. a.** $\frac{\log 30}{\log 3}$ **b.** 3.10