

Chapter 7

THE BINOMIAL EXPANSION

EXERCISE 7A

1 a
$$(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$$

b
$$(x-3)^3 = x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3 = x^3 - 9x^2 + 27x - 27$$

c
$$(3x-1)^3 = (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3 = 27x^3 - 27x^2 + 9x - 1$$

d
$$(2+x)^3 = 2^3 + 3(2)^2x + 3(2)x^2 + x^3 = 8 + 12x + 6x^2 + x^3$$

e
$$(2x+5)^3 = (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3 = 8x^3 + 60x^2 + 150x + 125$$

f
$$(3x-\frac{1}{3})^3 = (3x)^3 + 3(3x)^2(-\frac{1}{3}) + 3(3x)(-\frac{1}{3})^2 + (-\frac{1}{3})^3 = 27x^3 - 9x^2 + x - \frac{1}{27}$$

g
$$(2x+\frac{1}{x})^3 = (2x)^3 + 3(2x)^2(\frac{1}{x}) + 3(2x)(\frac{1}{x})^2 + (\frac{1}{x})^3 = 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$$

2 a
$$(1+x)^4 = 1^4 + 4(1)^3x + 6(1)^2x^2 + 4(1)x^3 + x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

b
$$(p-q)^4 = p^4 + 4p^3(-q) + 6p^2(-q)^2 + 4p(-q)^3 + (-q)^4 = p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$$

c
$$(x-2)^4 = x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16$$

d
$$(3-x)^4 = 3^4 + 4(3)^3(-x) + 6(3)^2(-x)^2 + 4(3)(-x)^3 + (-x)^4 = 81 - 108x + 54x^2 - 12x^3 + x^4$$

e
$$(1+2x)^4 = 1^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + (2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4$$

f
$$(2x+3)^4 = (2x)^4 + 4(2x)^3(3)^1 + 6(2x)^2(3)^2 + 4(2x)(3)^3 + (3)^4 = 16x^4 + 12 \times 8x^3 + 54 \times 4x^2 + 108 \times 2x + 81 = 16x^4 + 96x^3 + 216x^2 + 216x + 81$$

g
$$(x+\frac{1}{x})^4 = x^4 + 4x^3(\frac{1}{x}) + 6x^2(\frac{1}{x})^2 + 4x(\frac{1}{x})^3 + (\frac{1}{x})^4 = x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$

h
$$(2x-\frac{1}{x})^4 = (2x)^4 + 4(2x)^3(-\frac{1}{x}) + 6(2x)^2(-\frac{1}{x})^2 + 4(2x)(-\frac{1}{x})^3 + (-\frac{1}{x})^4 = 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$$

3 a
$$(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + 2^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

b
$$(x+1)^3 = x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3 = x^3 + 3x^2 + 3x + 1$$

c
$$(2+3)^3 = 2^3 + 3(2)^2x + 3(2)x^2 + x^3 = 8 + 12x + 6x^2 + x^3$$

d
$$(2x+5)^3 = (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3 = 8x^3 + 60x^2 + 150x + 125$$

e
$$(3x-\frac{1}{3})^3 = (3x)^3 + 3(3x)^2(-\frac{1}{3}) + 3(3x)(-\frac{1}{3})^2 + (-\frac{1}{3})^3 = 27x^3 - 9x^2 + x - \frac{1}{27}$$

f
$$(2x+\frac{1}{x})^3 = (2x)^3 + 3(2x)^2(\frac{1}{x}) + 3(2x)(\frac{1}{x})^2 + (\frac{1}{x})^3 = 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$$

g
$$(1+x)^4 = 1^4 + 4(1)^3x + 6(1)^2x^2 + 4(1)x^3 + x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

h
$$(p-q)^4 = p^4 + 4p^3(-q) + 6p^2(-q)^2 + 4p(-q)^3 + (-q)^4 = p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$$

i
$$(x-2)^4 = x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16$$

j
$$(3-x)^4 = 3^4 + 4(3)^3(-x) + 6(3)^2(-x)^2 + 4(3)(-x)^3 + (-x)^4 = 81 - 108x + 54x^2 - 12x^3 + x^4$$

k
$$(1+2x)^4 = 1^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + (2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4$$

l
$$(2x+3)^4 = (2x)^4 + 4(2x)^3(3)^1 + 6(2x)^2(3)^2 + 4(2x)(3)^3 + (3)^4 = 16x^4 + 12 \times 8x^3 + 54 \times 4x^2 + 108 \times 2x + 81 = 16x^4 + 96x^3 + 216x^2 + 216x + 81$$

m
$$(x+\frac{1}{x})^4 = x^4 + 4x^3(\frac{1}{x}) + 6x^2(\frac{1}{x})^2 + 4x(\frac{1}{x})^3 + (\frac{1}{x})^4 = x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$

n
$$(2x-\frac{1}{x})^4 = (2x)^4 + 4(2x)^3(-\frac{1}{x}) + 6(2x)^2(-\frac{1}{x})^2 + 4(2x)(-\frac{1}{x})^3 + (-\frac{1}{x})^4 = 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$$

o
$$(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + 2^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

b
$$(x-2y)^5 = x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + (-2y)^5 = x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$$

c
$$(1+2x)^5 = 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5 = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$$

d
$$\left(x - \frac{1}{x}\right)^5 = x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

4 a
$$\begin{array}{ccccccc} 1 & 5 & 10 & 10 & 5 & 1 & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array} \begin{array}{l} \text{the 5th row} \\ \text{the 6th row} \end{array}$$

b i
$$(x+2)^6 = x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

ii
$$\begin{aligned} (2x-1)^6 &= (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4 \\ &\quad + 6(2x)(-1)^5 + (-1)^6 \\ &= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1 \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1 \end{aligned}$$

iii
$$\begin{aligned} \left(x + \frac{1}{x}\right)^6 &= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6} \end{aligned}$$

5 a
$$\begin{aligned} (1+\sqrt{2})^3 &= (1)^3 + 3(1)^2(\sqrt{2}) + 3(1)(\sqrt{2})^2 + (\sqrt{2})^3 \\ &= 1 + 3\sqrt{2} + 3 \times 2 + 2 \times \sqrt{2} \\ &= 1 + 3\sqrt{2} + 6 + 2\sqrt{2} \\ &= 7 + 5\sqrt{2} \end{aligned}$$

b
$$\begin{aligned} (\sqrt{5}+2)^4 &= (\sqrt{5})^4 + 4(\sqrt{5})^3(2) + 6(\sqrt{5})^2(2)^2 + 4(\sqrt{5})(2)^3 + 2^4 \\ &= 25 + 8 \times 5\sqrt{5} + 24 \times 5 + 32\sqrt{5} + 16 \\ &= 25 + 40\sqrt{5} + 120 + 32\sqrt{5} + 16 \\ &= 161 + 72\sqrt{5} \end{aligned}$$

c
$$\begin{aligned} (2-\sqrt{2})^5 &= (2)^5 + 5(2)^4(-\sqrt{2}) + 10(2)^3(-\sqrt{2})^2 + 10(2)^2(-\sqrt{2})^3 + 5(2)^1(-\sqrt{2})^4 + (-\sqrt{2})^5 \\ &= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2} \\ &= 232 - 164\sqrt{2} \end{aligned}$$

6 a
$$\begin{aligned} (2+x)^6 &= (2)^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6 \\ &= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6 \end{aligned}$$

b
$$(2.01)^6 \text{ is obtained by letting } x = 0.01$$

$\therefore (2.01)^6 = 64 + 192 \times (0.01) + 240 \times (0.01)^2 + 160 \times (0.01)^3 + 60 \times (0.01)^4 + 12 \times (0.01)^5 + (0.01)^6$	64 1.92 0.024 0.000 16 0.000 000 6 0.000 000 001 2 + 0.000 000 000 001
	65.944 160 601 201

+ 0.000 000 000 001
65.944 160 601 201

6 a
$$(x+2)(x^2+1)^8$$

$$= (x+2) \left[(x^2)^8 + \binom{8}{1}(x^2)^7 1 + \binom{8}{2}(x^2)^6 1^2 + \dots + \binom{8}{6}(x^2)^2 1^6 + \binom{8}{7}(x^2)^1 1^7 + \binom{8}{8} 1^8 \right]$$

only terms which when multiplied give an x^5

$$\therefore \text{coefficient of } x^5 \text{ is } 1 \times \binom{8}{6} = \binom{8}{6} = 28.$$

b
$$(2-x)(3x+1)^9$$

$$= (2-x) \left[(3x)^9 + \binom{9}{1}(3x)^8 + \binom{9}{2}(3x)^7 + \binom{9}{3}(3x)^6 + \binom{9}{4}(3x)^5 + \dots \right]$$

only terms which when multiplied give an x^6

$$\therefore \text{coefficient of } x^6 \text{ is } 2 \times \binom{9}{3} \times 3^6 + (-1) \times \binom{9}{4} \times 3^5 = 2 \binom{9}{3} 3^6 - \binom{9}{4} 3^5 = 91854$$

7 In $(x^2y - 2y^2)^6$, $a = (x^2y)$, $b = (-2y^2)$ and $n = 6$.

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$= \binom{6}{r} (x^2y)^{6-r} (-2y^2)^r$$

$$= \binom{6}{r} x^{12-2r} y^{6-r} (-2)^r y^{2r}$$

$$= \binom{6}{r} (-2)^r x^{12-2r} y^{6+r}$$

Since x and y are raised to the same power, $12 - 2r = 6 + r$

$$\therefore 3r = 6$$

$$\therefore r = 2$$

$$T_3 = \binom{6}{2} (-2)^2 x^8 y^8$$

$$= 60x^8 y^8$$

8 a $(1+x)^n$ has $T_3 = \binom{n}{2} 1^{n-2} x^2 = \binom{n}{2} x^2$ and $n \geq 2$

But this term is $36x^2 \therefore \binom{n}{2} = 36$

$$\therefore \frac{n(n-1)}{2} = 36$$

$$\therefore n(n-1) = 72$$

$$\therefore n^2 - n - 72 = 0$$

$$\therefore (n-9)(n+8) = 0$$

$$\therefore n = 9 \text{ or } -8$$

But $n \geq 2$, so $n = 9$

$$\text{and } T_4 = \binom{9}{3} 1^{n-3} x^3$$

$$= \binom{9}{3} x^3$$

$$= 84x^3$$

b $(1+kx)^n = 1^n + \binom{n}{1} 1^{n-1} (kx)^1 + \binom{n}{2} 1^{n-2} (kx)^2 + \dots$

$$= 1 + \binom{n}{1} kx + \binom{n}{2} k^2 x^2 + \dots$$

$$\therefore \binom{n}{1} k = -12 \text{ and } \binom{n}{2} k^2 = 60$$

$$\therefore nk = -12 \text{ and } \frac{n(n-1)}{2} k^2 = 60$$

$$\therefore n(n-1)k^2 = 120$$

But $k = -\frac{12}{n} \therefore n(n-1) \frac{144}{n^2} = 120$

$$\therefore 144(n-1) = 120n \quad \{n \geq 2\}$$

$$\therefore 144n - 120n = 144$$

$$\therefore 24n = 144$$

$$\therefore n = 6 \text{ and so } k = -2$$

9 $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ where $n = 10$, $a = (x^2)$, $b = \left(\frac{1}{ax}\right)$

$$= \binom{10}{r} (x^2)^{10-r} \left(\frac{1}{ax}\right)^r$$

$$= \binom{10}{r} x^{20-2r} \times \frac{1}{a^r x^r}$$

$$= \binom{10}{r} x^{20-3r} \times \frac{1}{a^r}$$

We let $20 - 3r = 11$

$$\therefore 3r = 9$$

$$\therefore r = 3$$

So, $\frac{\binom{10}{3}}{a^3} = 15$

and $T_4 = \binom{10}{3} x^{11} \times \frac{1}{a^3}$

$$= \frac{\binom{10}{3}}{a^3} x^{11}$$

$$\therefore a^3 = 8$$

$$\therefore a = 2$$

REVIEW SET 7

1 a $(x-2y)^3 = x^3 + 3x^2(-2y) + 3x(-2y)^2 + (-2y)^3$

$$= x^3 - 6x^2y + 12xy^2 - 8y^3$$

b $(3x+2)^4 = (3x)^4 + 4(3x)^3(2) + 6(3x)^2(2)^2 + 4(3x)(2)^3 + (2)^4$

$$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$$

2 In the expansion of $(2x+5)^6$, $a = (2x)$, $b = 5$, $n = 6$

$T_{r+1} = \binom{n}{r} a^{n-r} b^r$ For the coefficient of x^3 we let $6 - r = 3$

$$= \binom{6}{r} (2x)^{6-r} 5^r \therefore r = 3$$

$$= \binom{6}{r} 2^{6-r} x^{6-r} 5^r \text{ and } T_4 = \underbrace{\binom{6}{3} 2^3 5^3}_{\therefore \text{the coefficient is } \binom{6}{3} 2^3 5^3 = 20000} x^3$$

3 In the expansion of $\left(2x^2 - \frac{1}{x}\right)^6$, $a = (2x^2)$, $b = \left(-\frac{1}{x}\right)$, $n = 6$

$T_{r+1} = \binom{n}{r} a^{n-r} b^r$ For the constant term we let $12 - 3r = 0$

$$= \binom{6}{r} (2x^2)^{6-r} \left(-\frac{1}{x}\right)^r \therefore r = 4$$

$$= \binom{6}{r} 2^{6-r} x^{12-2r} (-1)^r x^{-r} \text{ and } T_5 = \underbrace{\binom{6}{4} 2^2 (-1)^4}_{\therefore \text{the constant term is } \binom{6}{4} 2^2 (-1)^4 = 60} x^0$$

$$= \binom{6}{r} 2^{6-r} (-1)^r x^{12-3r}$$

4 The sixth row of Pascal's triangle is 1 6 15 20 15 6 1
 $\therefore (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

a $(x-3)^6 = x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6$

$$= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$$

b
$$\left(1 + \frac{1}{x}\right)^6$$

$$= (1)^6 + 6(1)^5 \left(\frac{1}{x}\right) + 15(1)^4 \left(\frac{1}{x}\right)^2 + 20(1)^3 \left(\frac{1}{x}\right)^3 + 15(1)^2 \left(\frac{1}{x}\right)^4 + 6(1) \left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6$$

$$= 1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}$$