

Name:

KSP

IB Mathematics HL Year 1

Test Ch1

NON-Calculator Section

30 Minutes

29 Points

Please show your workings in the space provided:

1. (a) Consider the following sequence of equations.

$$1 \times 2 = \frac{1}{3}(1 \times 2 \times 3),$$

$$1 \times 2 + 2 \times 3 = \frac{1}{3}(2 \times 3 \times 4),$$

$$1 \times 2 + 2 \times 3 + 3 \times 4 = \frac{1}{3}(3 \times 4 \times 5),$$

.....

- (i) Formulate a conjecture for the n^{th} equation in the sequence.

$$\therefore 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{1}{3}(n)(n+1)(n+2)$$

- (ii) Verify your conjecture for $n = 4$.

$$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 = 2 + 6 + 12 + 20 = 40 \quad (2)$$

and $\frac{1}{3}(4)(5)(6) = 4 \cdot 5 \cdot 2 = 40 \checkmark$

- (b) A sequence of numbers has the n^{th} term given by $u_n = 2^n + 3$, $n \in \mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false.

(2)

$$u_1 = 2^1 + 3 = 5$$

$$u_2 = 2^2 + 3 = 7$$

$$u_3 = 2^3 + 3 = 11$$

$$u_4 = 2^4 + 3 = 19$$

$$u_5 = 2^5 + 3 = 35 \quad \text{Not prime.}$$

\therefore Bill's conjecture is false!

[Proved by induction]

2. Expand and simplify $\left(x^2 - \frac{2}{x}\right)^4$.

$$\begin{array}{cccc} & 1 & 2 & 1 \\ & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{array}$$

(Total 4 marks)

$$\begin{aligned}
 &= 1(x^2)^4 + 4(x^2)^3\left(-\frac{2}{x}\right) + 6(x^2)^2\left(-\frac{2}{x}\right)^2 + 4(x^2)\left(-\frac{2}{x}\right)^3 + 1\left(-\frac{2}{x}\right)^4 \\
 &= x^8 + 4x^6\left(-\frac{2}{x}\right) + 6x^4\left(\frac{4}{x^2}\right) + 4x^2\left(\frac{-8}{x^3}\right) + \frac{16}{x^4} \\
 &= x^8 + -8x^5 + 24x^2 + -\frac{32}{x} + \frac{16}{x^4} \\
 &= x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4} \quad \dots
 \end{aligned}$$

3. The common ratio of the terms in a geometric series is 2^x .

- (a) State the set of values of x for which the sum to infinity of the series exists.

$$S_{\infty} = \frac{u_1}{1-r} \quad -1 < r < 1$$

$-1 < 2^x < 1$

$\therefore x < 0$ ~~But $2^x \neq 0$~~
 $(2^x \text{ will never be negative})$

- (b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40.

(4)
(Total 6 marks)

$$\text{First term } u_1 = 35$$

$$S_{\infty} = 40 = \frac{35}{1-2^x}$$

$$1-2^x = \frac{35}{40}$$

$$1-2^x = \frac{7}{8}$$

$$2^x = \frac{8}{8} - \frac{7}{8}$$

$$2^x = \frac{1}{8}$$

$$2^x = 2^{-3}$$

$\therefore x = -3$

(hard for
them?)

4. Determine the first three terms in the expansion of $(1-2x)^5(1+x)^7$ in ascending powers of x .

(Total 5 marks)

$$(1-2x)^5 = 1^5 + \binom{5}{1}1^4(-2x)^1 + \binom{5}{2}1^3(-2x)^2 + \dots$$

$$= 1 + 5(-2x) + 10(-2x)^2 + \dots$$

Note:

$$\binom{5}{2} = \frac{5!}{(5-2)!2!} = 1 - 10x + 10(4x^2) + \dots$$

$$= \frac{5!}{3! \cdot 2!} = 1 - 10x + 40x^2 + \dots$$

$$= \frac{5 \cdot 4 \cdot 3!}{3! \cdot 2!}$$

$$= \frac{20}{2}$$

$$= 10$$

$$(1+x)^7 = 1^7 + \binom{7}{1}1^6(x)^1 + \binom{7}{2}1^5(x)^2 + \dots$$

$$= 1 + 7x + 21x^2 + \dots$$

Note:

$$\binom{7}{2} = \frac{7!}{5! \cdot 2!}$$

$$= \frac{7 \cdot 6 \cdot 5!}{5! \cdot 2!}$$

$$= \frac{42}{2}$$

$$= 21$$

$$\therefore (1-2x)^5(1+x)^7$$

$$= (1-10x+40x^2+\dots)(1+7x+21x^2+\dots)$$

$$= (1-10x+40x^2+\dots)1 + (1-10x+40x^2+\dots)7x + (1-10x+40x^2)21x^2 + \dots$$

$$= 1 - \underline{10x} + \underline{40x^2} + \underline{7x} - \underline{70x^2} + \cancel{740x^3} + \underline{21x^2} - \cancel{210x^3} + \cancel{4021x^4}$$

$$= 1 - 10x + 7x + 40x^2 + 21x^2 - 70x^2 + \dots$$

(first 3 will only be
constant + $x + x^2$)

$$= 1 - 3x - 9x^2 + \dots$$

6. (a) Approximate the sum of the geometric sequence 40, 20, 10, 5, ... for the first 10 terms.

(3)

$$S_{10} = \frac{a(1-r^{10})}{1-r}$$

$$= \frac{40(1-(\frac{1}{2})^{10})}{1-\frac{1}{2}} \quad (\frac{1}{2})^{10} = \frac{1}{2^{10}} \approx 0$$

$$\therefore \approx \frac{40}{0.5} \approx 80$$

- (b) Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

(7)
(Total 10 marks)

(Use the next page to show your proof if necessary)

$$\text{Let } P(n) = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$$

$$\therefore P(1) = a \text{ (LHS)}$$

$$P(1) = \frac{a(1-r^1)}{1-r}$$

$$= a \text{ (RHS)}$$

$\therefore P(1)$ is true.

Assume $P(k)$. That is, $P(k) = \frac{a(1-r^k)}{1-r}$ is true.

$$\text{Then } P(k+1) = \underbrace{a + ar + ar^2 + ar^3 + \dots + ar^{k-1}}_{P(k)} + ar^k + U_{k+1}$$

$$= \frac{a(1-r^k)}{1-r} + ar^{(k+1)-1}$$

$$= \frac{a(1-r^k)}{1-r} + ar^k \left(\frac{1-r}{1-r} \right)^{k+1}$$

$$= \frac{a(1-r^k)}{1-r} + \frac{ar^k(1-r)}{1-r}$$

$$\begin{aligned}
 &= \frac{a(1-r^k)}{1-r} + \frac{a(r^k - r^{k+1})}{1-r} \\
 &= a \left[\frac{(1-r^k) + (r^k - r^{k+1})}{1-r} \right] \\
 &= a \left[\cancel{1-r^k} + \cancel{r^k} - r^{k+1} \right] \\
 &= \frac{a(1-r^{k+1})}{1-r} \\
 &= P(k+1)
 \end{aligned}$$

Since $P(1)$ is true, assuming $P(b)$ is true we have shown $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Q.E.D