SEQUENCES AND SERIES

A **number sequence** is a set of numbers defined by a rule which is valid for positive integers. Often, the rule is a formula for the **general term** or **nth term** of the sequence.

Arithmetic Sequences

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$ for all n, where d is a constant called the **common difference.**

For an arithmetic sequence with first term u_1 and common difference d, $u_n = u_1 + (n-1)\,d$.

Geometric Sequences

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant.

$\dfrac{u_{n+1}}{u_n} = r$ for all n, where r is a constant called the **common ratio.**

For a geometric sequence with first term u_1 and common ratio r, $u_n = u_1 r^{n-1}$.

Series

A **series** is the addition of the terms of a sequence. Given a series which includes the first n terms of a sequence, its sum is $S_n = u_1 + u_2 + ... + u_n$.

Using **sigma notation** we can write

$u_1 + u_2 + u_3 + ... + u_n$ as $\sum\limits_{k=1}^{n} u_k$.

For an **arithmetic series**, $S_n = \frac{n}{2}(u_1 + u_n)$.

For a **geometric series**, $S_n = \dfrac{u_1(r^n - 1)}{r-1}$.

The sum of an **infinite geometric series** is

$S = \dfrac{u_1}{1-r}$ provided $|r| < 1$.

If $|r| > 1$ the series is **divergent.**

For compound interest problems we have a geometric sequence. If the interest rate is $i\%$ per time period then the common ratio is $\left(1 + \frac{i}{100}\right)$ and the number of compounding periods is n.

EXPONENTIALS AND LOGARITHMS

Exponential and logarithmic functions are inverses of each other. The graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.

Index or Exponent Laws	
$a^x \times a^y = a^{x+y}$	$a^{-x} = \dfrac{1}{a^x}$ and $\dfrac{1}{a^{-x}} = a^x$
$\dfrac{a^x}{a^y} = a^{x-y}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$
$(a^x)^y = a^{xy}$	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
$a^0 = 1 \;\; (a \neq 0)$	

If $a^x = a^k$ then $x = k$. So, if the base numbers are the same, we can **equate indices.**

If $b = a^x$, $a \neq 1$, $a > 0$, we say that x is the **logarithm** of b in base a, and that $b = a^x \Leftrightarrow x = \log_a b$, $b > 0$.

The **natural logarithm** is the logarithm in base e. $\ln x \equiv \log_e x$

Logarithm Laws	
Base a	**Base e**
$\log_a xy = \log_a x + \log_a y$	$\ln xy = \ln x + \ln y$
$\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$	$\ln \left(\dfrac{x}{y}\right) = \ln x - \ln y$
$\log_a x^y = y \log_a x$	$\ln x^y = y \ln x$
$\log_a 1 = 0$	$\ln 1 = 0$
$\log_a a = 1$	$\ln e = 1$

To change the base of a logarithm, use the rule $\log_b x = \dfrac{\log_c x}{\log_c b}$.

$x = \log_a a^x$ and $x = a^{\log_a x}$ provided $x > 0$.

THE BINOMIAL THEOREM

$a + b$ is called a **binomial** as it contains two terms.

Any expression of the form $(a+b)^n$ is called a **power of a binomial.**

The **general binomial expansion** is

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + ... + \binom{n}{r} a^{n-r}b^r + ... + \binom{n}{n} b^n$$

where $\binom{n}{r}$ is the binomial coefficient of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, ..., n$.

The **general term** in the binomial expansion is

$T_{r+1} = \binom{n}{r} a^{n-r}b^r$, so $(a+b)^n = \sum\limits_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$.

You should know how to calculate binomial coefficients from Pascal's triangle and using your calculator.

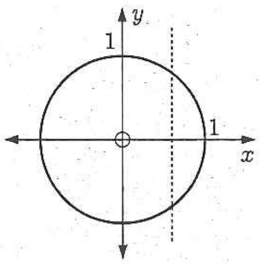
TOPIC 2: FUNCTIONS AND EQUATIONS

FUNCTIONS $f : x \mapsto f(x)$ OR $y = f(x)$

A **relation** is any set of points on the Cartesian plane.

A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first member. For each value of x there is only one value of y or $f(x)$. We sometimes refer to y or $f(x)$ as the **image value** of x.

We **test for functions** using the vertical line test. A graph is a function if no vertical line intersects the graph more than once.



For example, a circle such as $x^2 + y^2 = 1$ has a graph which is not a function.

The **domain** of a function is the set of values that x can take.

To find the domain of a function, remember that we cannot:

- divide by zero
- take the square root of a negative number
- take the logarithm of a non-positive number.

The **range** of a function is the set of values that y or $f(x)$ can take.

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g is $f \circ g : x \mapsto f(g(x))$.

In general, $f(g(x)) \neq g(f(x))$, so $f \circ g \neq g \circ f$.

The **identity function** is $f(x) = x$.

If $y = f(x)$ has an **inverse function** $y = f^{-1}(x)$, then the inverse function:

- must satisfy the vertical line test
- is a reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- has range equal to the domain of $f(x)$
- has domain equal to the range of $f(x)$.

The **reciprocal function** is $f(x) = \frac{1}{x}$, $x \neq 0$.

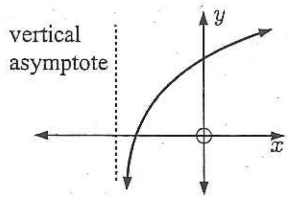
The reciprocal function is a **self-inverse function,** as $f^{-1}(x) = f(x) = \frac{1}{x}$.

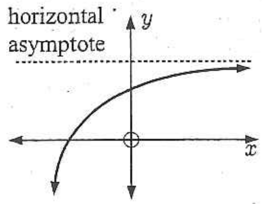
GRAPHS OF FUNCTIONS

The **x-intercepts** of a function are the values of x for which $y = 0$. They are the **zeros** of the function.

The **y-intercept** of a function is the value of y when $x = 0$.

An **asymptote** is a line that the graph *approaches* or begins to look like as it tends to infinity in a particular direction.





To find vertical asymptotes, look for values of x for which the function is undefined:

- if $y = \dfrac{f(x)}{g(x)}$ find where $g(x) = 0$
- if $y = \log_a(f(x))$ find where $f(x) = 0$.

To find horizontal asymptotes, consider the behaviour as $x \to \pm\infty$.

Transformations of graphs

- $y = f(x) + b$ **translates** $y = f(x)$ vertically b units.
- $y = f(x - a)$ **translates** $y = f(x)$ horizontally a units.
- $y = f(x - a) + b$ **translates** $y = f(x)$ by the vector $\binom{a}{b}$.
- $y = pf(x), p > 0$ is a **vertical stretch** of $y = f(x)$ with dilation factor p.
- $y = f\left(\dfrac{x}{q}\right)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with dilation factor q.
- $y = -f(x)$ is a **reflection** of $y = f(x)$ in the x-axis.
- $y = f(-x)$ is a **reflection** of $y = f(x)$ in the y-axis.
- $y = f^{-1}(x)$ is a **reflection** of $y = f(x)$ in the line $y = x$.

LINEAR FUNCTIONS

A **linear function** has the form $f(x) = ax + b$, $a \neq 0$.

Its graph is a straight line with gradient a and y-intercept b.

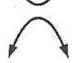
Perpendicular lines have gradients which are the negative reciprocals of each other.

QUADRATIC FUNCTIONS

A **quadratic function** has the form $f(x) = ax^2 + bx + c$, $a \neq 0$.

The graph is a parabola with the following properties:

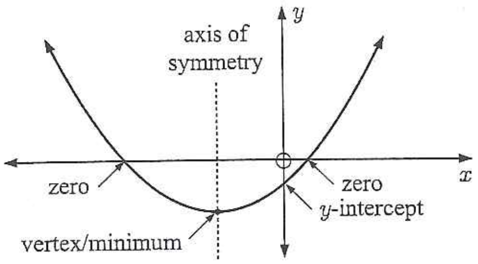
- it is *concave up* if $a > 0$ 

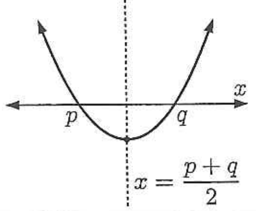
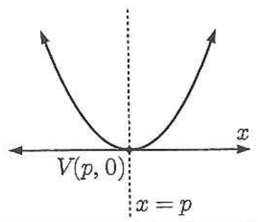
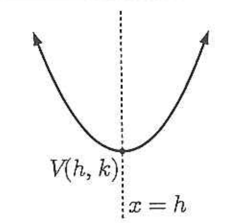
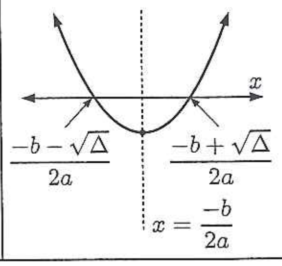
  and *concave down* if $a < 0$ 

- its axis of symmetry is $x = \dfrac{-b}{2a}$
- its vertex is at $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$.

A quadratic function written in the form:

- $f(x) = a(x-h)^2 + k$ has vertex (h, k)
- $f(x) = a(x-p)(x-q)$ has x-intercepts p and q.



$y = a(x-p)(x-q)$ x-intercepts p, q axis of symmetry $x = \dfrac{p+q}{2}$	
$y = a(x-p)^2$ touches x-axis at p vertex $(p, 0)$ axis of symmetry $x = p$	
$y = a(x-h)^2 + k$ vertex (h, k) axis of symmetry $x = h$	
$y = ax^2 + bx + c$ axis of symmetry $x = \dfrac{-b}{2a}$ x-intercepts $\dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac \geqslant 0$	

QUADRATIC EQUATIONS

A quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, can be solved by:

- factorisation
- completing the square
- the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The **discriminant** of the quadratic equation is $\Delta = b^2 - 4ac$.

The quadratic equation has:

- *no real solutions* if $\Delta < 0$
- *one real (repeated) solution* if $\Delta = 0$
- *two real solutions* if $\Delta > 0$.

The number of solutions indicates whether the graph of the corresponding quadratic function *does not meet*, *touches*, or *cuts* the x-axis.

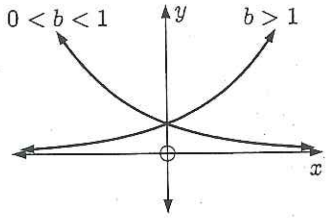
A quadratic function is **positive definite** if $a > 0$ and $\Delta < 0$.
A quadratic function is **negative definite** if $a < 0$ and $\Delta < 0$.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The simplest **exponential function** is $f(x) = b^x$, $b > 0$.

If $b > 1$ we have *growth*.

If $0 < b < 1$ we have *decay*.

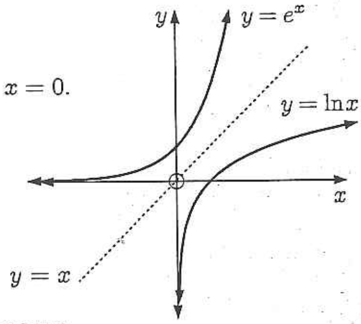


The graph of $y = b^x$ has the horizontal asymptote $y = 0$.

$b^x > 0$ and $b^{-x} > 0$ for all $x \in \mathbb{R}$.

The inverse function of $f(x) = b^x$ is the **logarithmic function** $f^{-1}(x) = \log_b x$, $x > 0$.

The graph of $y = \log_b x$ has the vertical asymptote $x = 0$.



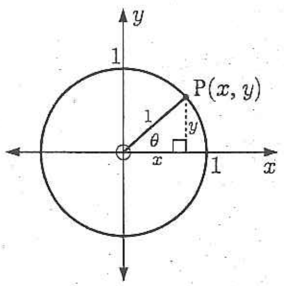
EXPONENTIAL EQUATIONS

- If we can make the base numbers the same then we can equate indices.
  So, if $a^x = a^k$ then $x = k$.
- If the bases cannot be made the same then we take the logarithm of both sides.
  For example: If $2^x = 30$
  then $\log(2^x) = \log 30$
  $\therefore \; x \log 2 = \log 30$
  $\therefore \; x = \dfrac{\log 30}{\log 2} \approx 4.91$

TOPIC 3: CIRCULAR FUNCTIONS AND TRIGONOMETRY

The **unit circle** is the circle centred at the origin O and with radius 1 unit.

The coordinates of any point P on the unit circle, where the angle θ is made by [OP] and the positive x-axis, are $(\cos\theta, \sin\theta)$.



θ is **positive** when measured in an **anticlockwise** direction from the positive x-axis.

From the unit circle we can see that:

- $\cos^2\theta + \sin^2\theta = 1$
- $-1 \leqslant \cos\theta \leqslant 1$ and $-1 \leqslant \sin\theta \leqslant 1$ for all θ
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ provided $\cos\theta \neq 0$.

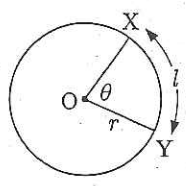
2π radians is equivalent to 360°.

To convert from degrees to radians, multiply by $\frac{\pi}{180}$.

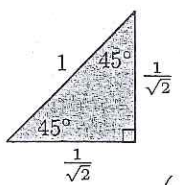
To convert from radians to degrees, multiply by $\frac{180}{\pi}$.

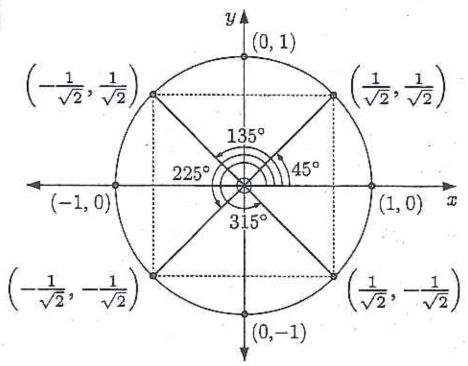
For θ in radians:

- the length of an arc of radius r and angle θ is $l = \theta r$
- the area of a sector of radius r and angle θ is $A = \frac{1}{2}\theta r^2$.

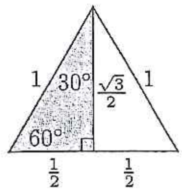


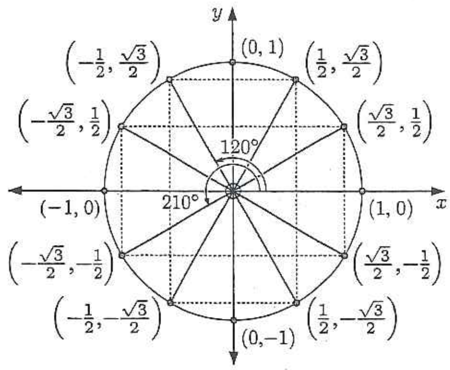
MULTIPLES OF 45° OR $\frac{\pi}{4}$



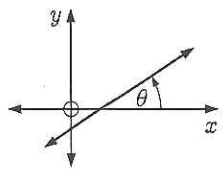


MULTIPLES OF 30° OR $\frac{\pi}{6}$





If a straight line makes an angle of θ with the positive x-axis, then its gradient is $m = \tan\theta$.



NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

For the triangle alongside:

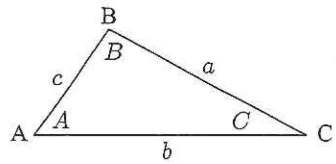
Area formula

Area $= \frac{1}{2}ab\sin C$

Cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A$

Sine rule $\quad \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$



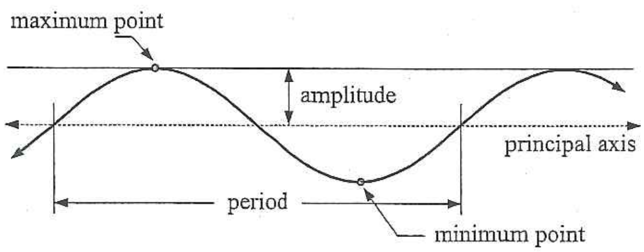
If you have the choice of rules to use, use the cosine rule to avoid the **ambiguous case**.

PERIODIC FUNCTIONS

A **periodic function** is one which repeats itself over and over in a horizontal direction.

The **period** of a periodic function is the length of one cycle.

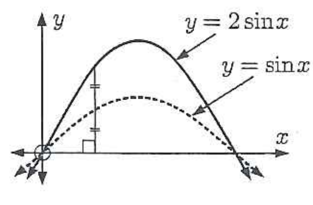
For example, a **wave** oscillates about a horizontal line called the **principal axis**.



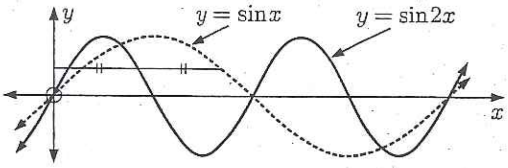
The **amplitude** is the distance between a maximum or minimum point and the principal axis.

TRANSFORMING $y = \sin x$

- $y = a\sin x$ is a **vertical stretch** of $y = \sin x$ with dilation factor a.



- $y = \sin bx$ is a **horizontal stretch** of $y = \sin x$ with dilation factor $\frac{1}{b}$.



- $y = \sin(x - c)$ is a **horizontal translation** of $y = \sin x$ through c units.
- $y = \sin x + d$ is a **vertical translation** of $y = \sin x$ through d units.

THE GENERAL SINE FUNCTION

If we begin with $y = \sin x$, we can perform transformations to produce the **general sine function** $f(x) = a \sin b(x - c) + d$.

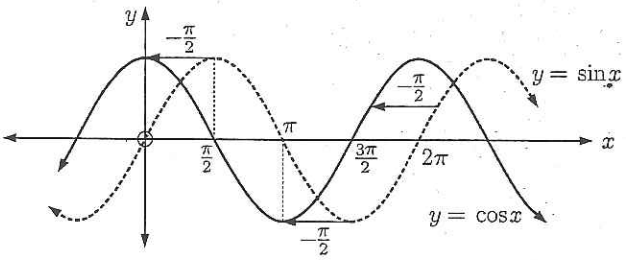
We have a vertical stretch with factor a and a horizontal stretch with factor $\frac{1}{b}$, followed by a translation with vector $\begin{pmatrix} c \\ d \end{pmatrix}$.

The general sine function has the following properties:

- the **amplitude** is $|a|$
- the **principal axis** is $y = d$
- the **period** is $\frac{2\pi}{b}$.

THE COSINE FUNCTION

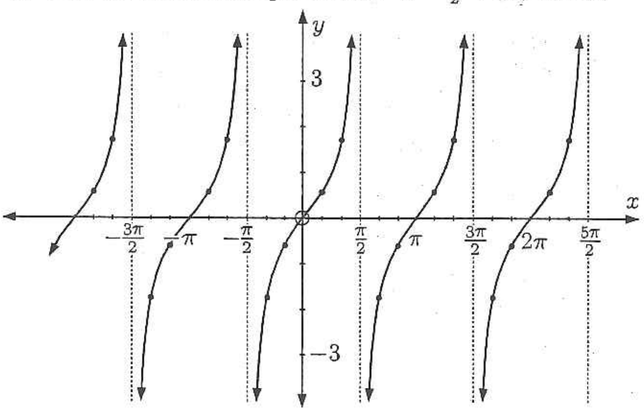
Since $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, the graph of $y = \cos x$ is a horizontal translation of $y = \sin x$, $\frac{\pi}{2}$ units to the left.



THE TANGENT FUNCTION

$y = \tan x = \dfrac{\sin x}{\cos x}$ is undefined when $\cos x = 0$.

$\therefore$ there are vertical asymptotes when $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.



TRIGONOMETRIC IDENTITIES

$\cos^2\theta + \sin^2\theta = 1$

$\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$ for all $k \in \mathbb{Z}$

$\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$ and $\tan(-\theta) = -\tan\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos(\pi - \theta) = -\cos\theta$ and $\sin(\pi - \theta) = \sin\theta$

$\sin 2A = 2\sin A\cos A$

$$\cos 2A = \begin{cases} \cos^2 A - \sin^2 A \\ 1 - 2\sin^2 A \\ 2\cos^2 A - 1 \end{cases}$$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

TRIGONOMETRIC EQUATIONS

To solve trigonometric equations we can either use graphs from technology, or algebraic methods involving the trigonometric identities. In either case we must make sure to include all solutions on the specified domain.

We need to use the inverse trigonometric functions to invert sin, cos and tan.

Equation	*Function*	*Domain*	*Range*
$\sin x = k$	$x = \sin^{-1} k$	$-1 \leqslant k \leqslant 1$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$
$\cos x = k$	$x = \cos^{-1} k$	$-1 \leqslant k \leqslant 1$	$0 \leqslant x \leqslant \pi$
$\tan x = k$	$x = \tan^{-1} k$	$k \in \mathbb{R}$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

The ranges of these functions are important because our calculator will only give us the one answer in the range. Sometimes other solutions may also be possible. For example, when using $\sin^{-1}$ our calculator will always give us an acute angle answer, but the obtuse angle with the same sine may also be valid.

An equation of the form $a\sin x = b\cos x$ can always be solved as $\tan x = \frac{b}{a}$.

TOPIC 4: MATRICES

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

If a matrix has m rows and n columns, its **order** is $m \times n$.

Two matrices are **equal** if they have the same order and the elements in corresponding positions are equal.

MATRIX OPERATIONS

To **add** two matrices of the same order, we **add** the elements in corresponding positions.

To **subtract** two matrices of the same order, we **subtract** the elements in corresponding positions.

A **zero matrix O** is a matrix in which all elements are zero.

To multiply a matrix by a **scalar**, we multiply every element by that scalar.

The **negative** of matrix **A** is $-\mathbf{A}$ or $-1\mathbf{A}$. It is obtained by reversing the signs of every element in **A**.

If **A**, **B** and **C** are matrices of the same order then:

- $\mathbf{A}+\mathbf{B}$ is a matrix of the same order
- $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
- $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
- $\mathbf{A}+\mathbf{O}=\mathbf{O}+\mathbf{A}=\mathbf{A}$
- $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}$
- a half of **A** is $\frac{1}{2}\mathbf{A}$, not $\dfrac{\mathbf{A}}{2}$

MATRIX MULTIPLICATION

We can only **multiply** matrix **A** by matrix **B** if the number of columns in **A** equals the number of rows in **B**.

If **A** is $m \times p$ and **B** is $p \times n$ then the product matrix **AB** is $m \times n$. The element in the rth row and cth column of **AB** is the sum of the products of the elements in the rth row of **A** with the corresponding elements in the cth column of **B**.

For example:

If $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

then $\mathbf{AB}=\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$.

If $\mathbf{C}=\underset{2\times 3}{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}$ and $\mathbf{D}=\underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}$

then $\mathbf{CD}=\underset{2\times 1}{\begin{pmatrix} ax+by+cz \\ dx+ey+fz \end{pmatrix}}$.

In general, $\mathbf{AB} \neq \mathbf{BA}$.

$\mathbf{A}+\mathbf{O}=\mathbf{O}+\mathbf{A}=\mathbf{A}$ for all matrices **A** of the same order.

An **identity matrix I** is a square matrix with 1s along the leading diagonal and zeros everywhere else.

$\mathbf{AI}=\mathbf{IA}=\mathbf{A}$ for all square matrices **A** of the same order.

$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$

MATRIX DETERMINANTS

The **determinant** of the matrix $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det\mathbf{A}$ or $|\mathbf{A}|=ad-bc$.

The determinant of the matrix $\mathbf{A}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$

is $|\mathbf{A}|=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}=a\begin{vmatrix} e & f \\ h & k \end{vmatrix}-b\begin{vmatrix} d & f \\ g & k \end{vmatrix}+c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

MATRIX INVERSES

The **inverse** of square matrix **A** is the matrix $\mathbf{A}^{-1}$ such that $\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$.

If $|\mathbf{A}|=0$ then $\mathbf{A}^{-1}$ does not exist and **A** is **singular**.

If $|\mathbf{A}|\neq 0$ then **A** is **invertible**.

If $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{A}^{-1}=\dfrac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

You should be able to find $|\mathbf{A}|$ and $\mathbf{A}^{-1}$ for 3×3 matrices using your calculator.

$|\mathbf{AB}|=|\mathbf{A}|\,|\mathbf{B}|$ for all square matrices **A** and **B** of equal size.

If **A** and **B** are invertible then $\left(\mathbf{A}^{-1}\right)^{-1}=\mathbf{A}$ and $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Any system of linear equations can be written as a matrix equation of the form $\mathbf{AX}=\mathbf{B}$ where **A** is a square matrix of coefficients, **X** is a column matrix of unknowns, and **B** is a column matrix of constants.

If **A** is **invertible** then the unique solution to the system is $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$.

TOPIC 5: VECTORS

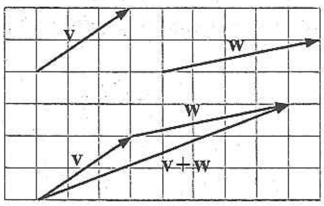
A **scalar** is a quantity which has only size or **magnitude**.

A **vector** is a quantity with both **magnitude** and **direction**.

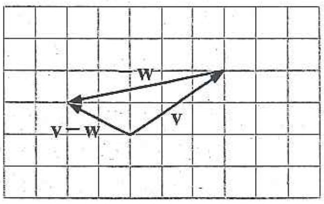
Two vectors are **equal** if they have the same magnitude *and* direction.

VECTORS IN GEOMETRIC FORM

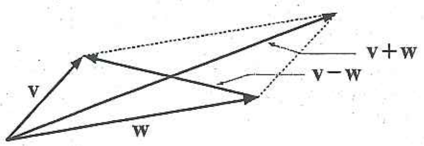
Given two vectors **v** and **w**, to find $\mathbf{v}+\mathbf{w}$ we first start by drawing **v**, and then from the end of **v** we draw **w**. The vector $\mathbf{v}+\mathbf{w}$ starts at the beginning of **v** and ends at the end of **w**.



To find $\mathbf{v}-\mathbf{w}$ we find $\mathbf{v}+(-\mathbf{w})$. We start by drawing **v**, then from the end of **v** we draw $-\mathbf{w}$.



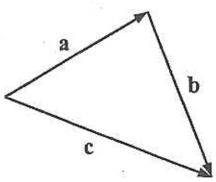
So, $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are represented by the diagonals of a parallelogram with defining vectors **v** and **w**.



The **zero vector 0**, is a vector of length 0. It is the only vector with no direction.

In the diagram alongside,

$$\mathbf{a}+\mathbf{b}=\mathbf{c}$$
$$\text{or } \mathbf{a}+\mathbf{b}-\mathbf{c}=\mathbf{0}.$$



VECTORS IN COMPONENT FORM

The basic unit vectors with magnitude 1 are:

in 2-D: $\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$

in 3-D: $\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

The **zero vector 0** is $\begin{pmatrix}0\\0\end{pmatrix}$ in 2-D and $\begin{pmatrix}0\\0\\0\end{pmatrix}$ in 3-D.

The general 2-D vector $\mathbf{v}=\begin{pmatrix}v_1\\v_2\end{pmatrix}=v_1\mathbf{i}+v_2\mathbf{j}$

The general 3-D vector $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$.

In examinations, scalars are written in italics, for example a, and vectors are written in bold type, for example **a**. On paper you should write vector **a** as $\overrightarrow{a}$.

You should understand the following for vectors in component form:

- vector equality
- vector addition
  - $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
  - $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$
  - $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$
  - $\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$
- vector subtraction $\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$
- multiplication by a scalar k to produce vector $k\mathbf{v}$ which is parallel to **v**
  - if $k>0$, $k\mathbf{a}$ and **a** have the same direction
  - if $k<0$, $k\mathbf{a}$ and **a** have opposite directions
  - if $k=0$, $k\mathbf{a}=\mathbf{0}$
- the magnitude of vector **v**, $|\mathbf{v}|=\sqrt{v_1^2+v_2^2+v_3^2}$
- the distance between two points in space is the magnitude of the vector which joins them
- $|k\mathbf{a}|=|k|\,|\mathbf{a}|$

POSITION VECTORS IN 2-D

The **position vector** of $A(x, y)$ is $\overrightarrow{OA}$ or $\mathbf{a}=\begin{pmatrix}x\\y\end{pmatrix}$.

The position vector of B relative to A is $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\mathbf{b}-\mathbf{a}$.

Given $A(x_1, y_1)$ and $B(x_2, y_2)$:

- the length of $\overrightarrow{AB}$ is $AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
- the midpoint of $\overrightarrow{AB}$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

POSITION VECTORS IN 3-D

The **position vector** of $A(x, y, z)$ is $\overrightarrow{OA}$ or $\mathbf{a}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Given $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the length of $\overrightarrow{AB}$ is
$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
- the midpoint of $\overrightarrow{AB}$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

PROPERTIES OF VECTORS

If X divides [AB] in the ratio $a : b$ then $\overrightarrow{AX} : \overrightarrow{XB} = a : b$.

A, B and C are **collinear** if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\mathbf{a} = k\mathbf{b}$ for some constant $k \neq 0$.

The unit vector in the direction of $\mathbf{a}$ is $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$.

A vector $\mathbf{b}$ of length k in the same direction as $\mathbf{a}$ is $\mathbf{b} = \dfrac{k}{|\mathbf{a}|}\mathbf{a}$.

A vector $\mathbf{b}$ of length k which is *parallel* to $\mathbf{a}$ could be $\mathbf{b} = \pm\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.

The scalar or dot product of two vectors

$\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$ where θ is the angle between the vectors.

$\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$

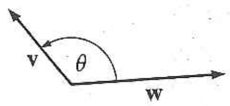
For non-zero vectors $\mathbf{v}$ and $\mathbf{w}$:

- $\mathbf{v} \perp \mathbf{w} \Leftrightarrow \mathbf{v} \bullet \mathbf{w} = 0$
- $\mathbf{v} \parallel \mathbf{w} \Leftrightarrow \mathbf{v} = k\mathbf{w}$ or $\mathbf{v} \bullet \mathbf{w} = \pm|\mathbf{v}||\mathbf{w}|$

The angle θ between vectors $\mathbf{v}$ and $\mathbf{w}$ emanating from the same point is given by $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$.

If $\mathbf{v} \bullet \mathbf{w} > 0$ then θ is acute.

If $\mathbf{v} \bullet \mathbf{w} < 0$ then θ is obtuse.

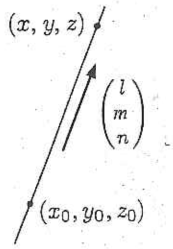


LINES

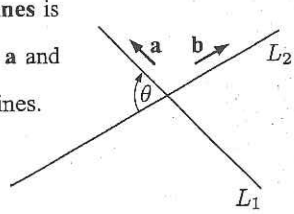
The **vector equation of a line** is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{a}$ is the position vector of any point on the line, $\mathbf{b}$ is a vector parallel to the line, and $t \in \mathbb{R}$.

For example, if an object has initial position vector $\mathbf{a}$ and moves with constant velocity $\mathbf{b}$, its position at time t is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ for $t \geqslant 0$.

If $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$, the **parametric form** for the equation of a line is $x = x_0 + tl$, $y = y_0 + tm$, $z = z_0 + tn$



The **acute angle θ between two lines** is given by $\cos\theta = \dfrac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$ where $\mathbf{a}$ and $\mathbf{b}$ are the direction vectors of the lines.



Lines are:

- **parallel** if their direction vectors are parallel: $\mathbf{a} = k\mathbf{b}$
- **coincident** if they are parallel and have a common point
- **intersecting** if you can solve them simultaneously to find a unique common point that fits both equations
- **skew** if they are not parallel and do not have a point of intersection; there are no solutions when the equations are solved simultaneously.

The shortest distance from point A to a line with direction vector $\mathbf{b}$ occurs at the point P on the line such that $\overrightarrow{AP}$ is perpendicular to $\mathbf{b}$.