The following portfolio investigates the sum of infinite sequences t_n, where

$$t_0 = 1, \quad t_1 = \frac{(x \ln a)}{1}, \quad t_2 = \frac{(x \ln a)^2}{2 \times 1}, \quad t_3 = \frac{(x \ln a)^3}{3 \times 2 \times 1} \dots, \quad t_n = \frac{(x \ln a)^n}{n!}$$

From this sequence it is apparent that $a > 0$, as ln0 or ln or any negative number would make the value of the term undefined. In this sequence the power of the numerator is dependant on the term number (n)

I considered the following sequence of terms when $x = 1$ and $a = 2$.

$$1, \quad \frac{(\ln 2)}{1}, \quad \frac{(\ln 2)^2}{2 \times 1}, \quad \frac{(\ln 2)^3}{3 \times 2 \times 1} \dots$$

Using Microsoft Excel, I found the sum S_n of the first n terms of the above sequence for $0 \le n \le 10$ which is shown in the table below

n	S_n
0	1.000000
1	1.693147
2	1.933374
3	1.988878
4	1.998496
5	1.999829
6	1.999983
7	1.999999
8	2.000000
9	2.000000
10	2.000000

The values form the table were then represented on a graph also by using Microsoft Excel in order to present the relation between S_n and n.
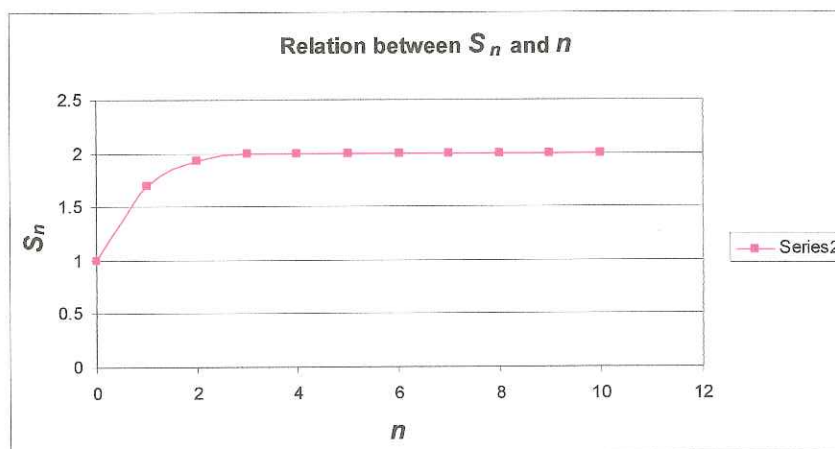
$n \in N$ so graph should be plotted as individual points

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From the above plot I noticed that S_n has an asymptote as it is approaching a unique value. The asymptote for S_n when $a = 2$ is 2. This suggests that as $n \to \infty$, the value of $S_n \to 2$.

I then considered another sequence of terms where $x = 1$ and $a = 3$

$$1, \; \frac{(\ln 3)}{1}, \; \frac{(\ln 3)^2}{2 \times 1}, \; \frac{(\ln 3)^3}{3 \times 2 \times 1} \cdots$$

Using Microsoft Excel, I found the sum S_n of the first n terms of the above sequence for $0 \le n \le 10$ which is shown in the table below

n	S_n
0	1.000000
1	2.098612
2	2.702087
3	2.923082
4	2.983779
5	2.997115
6	2.999557
7	2.999940
8	2.999993
9	2.999999
10	3.000000

The values form the table were then represented on a graph also by using Microsoft Excel in order to present the relation between S_n and n.
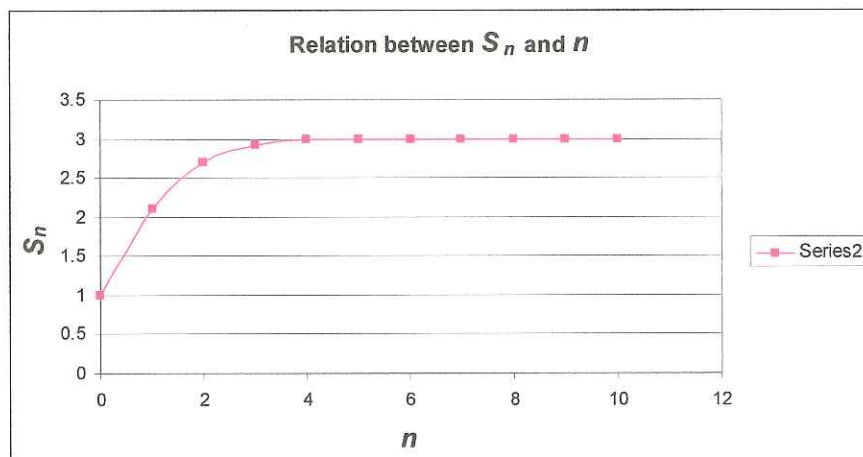
From the plot I noticed that S_n has an asymptote as it is approaching a unique value. The asymptote for S_n when $a = 3$ is 3. This suggests that as $n \to \infty$, the value of $S_n \to 3$.

I then considered a general sequence where $x = 1$.

$$1, \frac{(\ln a)}{1}, \frac{(\ln a)^2}{2 \times 1}, \frac{(\ln a)^3}{3 \times 2 \times 1} \cdots$$

I then calculated the sum S_n of the first n terms of this general sequence for $0 \leq n \leq 10$ for different values of a in order to find a general statement that represents the infinite sum of this general sequence.

When choosing the different numbers to test I took into account different types of numbers as I already tested natural numbers ($\mathbb{N}$). The different types of numbers I decided to test for this sequence were irrational numbers ($\mathbb{Q}'$) and larger natural numbers. It is important to note that ln0 and ln of any negative number would make the value of the term undefined.

I then considered the sequence where $x = 1$ and $a = \pi$

$$1, \frac{(\ln \pi)}{1}, \frac{(\ln \pi)^2}{2 \times 1}, \frac{(\ln \pi)^3}{3 \times 2 \times 1} \cdots$$

Using Microsoft Excel, I found the sum S_n of the first n terms of the above sequence for $0 \leq n \leq 10$ which is shown in the table below

n	S_n
0	1.000000
1	2.144730
2	2.799933
3	3.049943
4	3.121492
5	3.137873
6	3.140998
7	3.141509
8	3.141582
9	3.141591
10	3.141593

The values from the table were then represented on a graph also by using Microsoft Excel in order to present the relation between S_n and n.

Relation between S_n and n
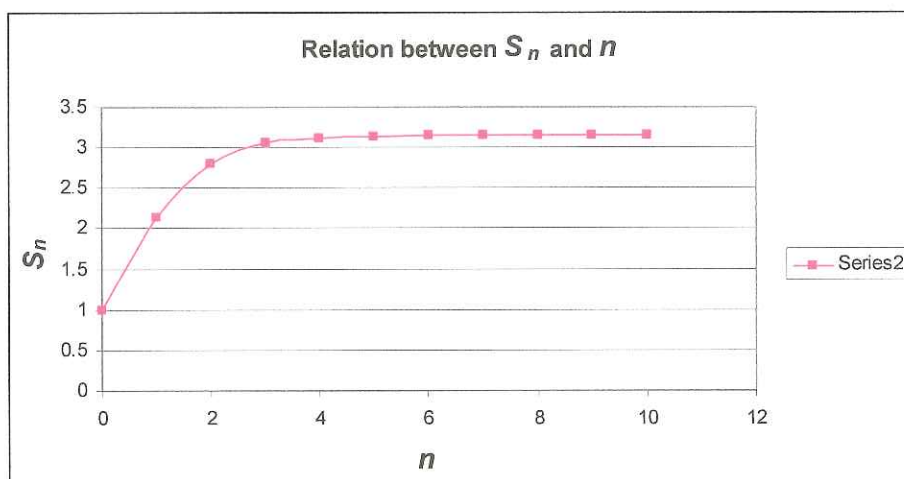
From the plot, I noticed that S_n has an asymptote which is π. This is the same as the value for a. This suggests that as $n \to \infty$, the value of $S_n \to \pi$.

The next sequence I considered was when $x = 1$ and $a = 80$. This is for the testing of large natural numbers.

$$1, \frac{(\ln 80)}{1}, \frac{(\ln 80)^2}{2 \times 1}, \frac{(\ln 80)^3}{3 \times 2 \times 1} \ldots$$

The sum S_n of the first n terms of the above sequence for $0 \le n \le 10$ is shown in the table bellow

n	S_n
0	1.000000
1	5.382027
2	14.983110
3	29.007170
4	44.370620
5	57.835230
6	67.668940
7	73.824890
8	77.196820
9	78.838590
10	79.558020

The values from the table were then represented on a graph in order to present the relation between S_n and n.

Relation between S_n and n
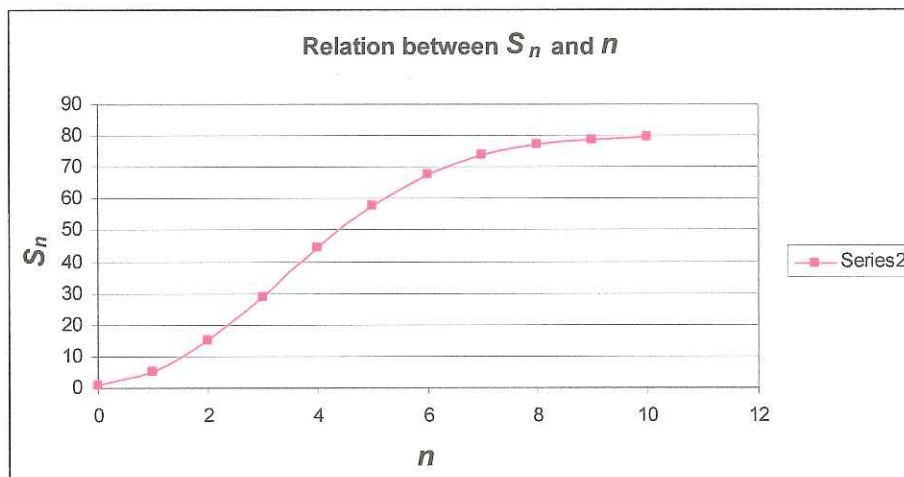
From this scatter-plot, one can see that the value of S_n approaches 80. This suggests that as $n \to \infty$ the value of $S_n \to 80$.

From my observations in these investigations one can concluded that the infinite sum from the sequence $t_n = \dfrac{(x \ln a)^n}{n!}$ will always be equal to the number substituted for a. This general statement applies when $a > 0$ because any number less than 0 will give an undefined value for the term. Also for this statement to be true $x = 1$. ✓
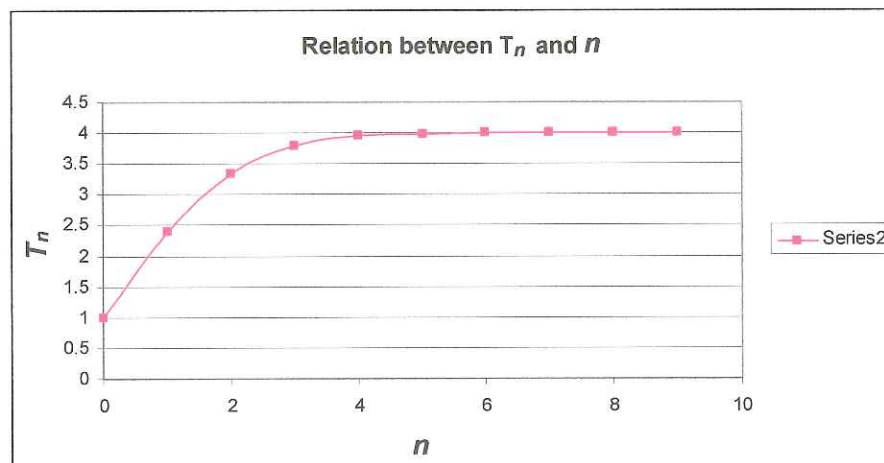
For the last sections of the investigation, I investigated the sum of the infinite sequence t_n for various values of a and x.

I considered the following sequence

$$t_0 = 1, \ t_1 = \frac{(x \ln 2)}{1}, \ t_2 = \frac{(x \ln 2)^2}{2 \times 1}, \ t_3 = \frac{(x \ln 2)^3}{3 \times 2 \times 1} \ \ldots, \ t_9 = \frac{(x \ln 2)^9}{9!}$$

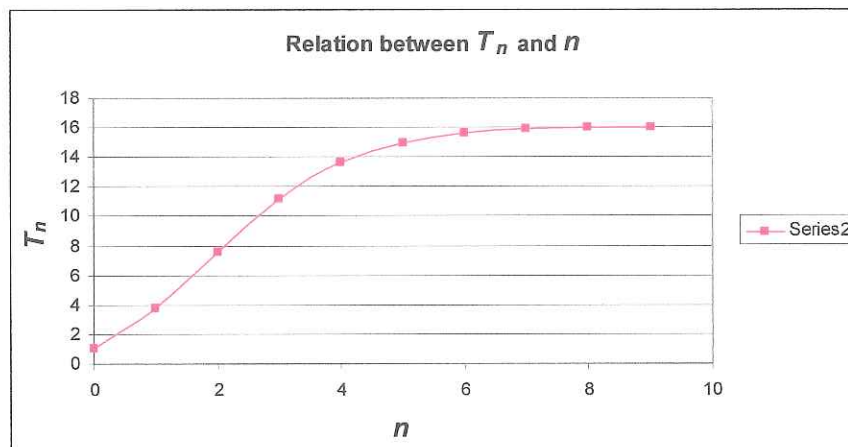As $a = 2$, I calculated $T_9(2, x)$ for various positive values of x.

what does $T_9(2,n)$ represent

So, as $a = 2$ and $x = 2$ we get the following results.

Relation between T_n and n

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From the plot $x = 2$, one observes that when $n \to \infty$ the value of $T_n \to 4$.

Next, I considered when $a = 2$ and $x = 4$. The results are represented on the following graph.

**Relation between T_n and n**

This plot shows that when $n \to \infty$, the value of $T_n \to 16$.

More values for x were tested, then the results were presented on a table and a graph shown below. These show the relation between $T_9(2, x)$ and x.

x	$T_9(2, x)$
1.5	2.828427
2	3.999992
3	7.999488
3.141593	8.821124
4	15.99019
5	31.90092
6	63.33107

The above table is represented in the graph below.

Relation between x and $T_9(2, x)$
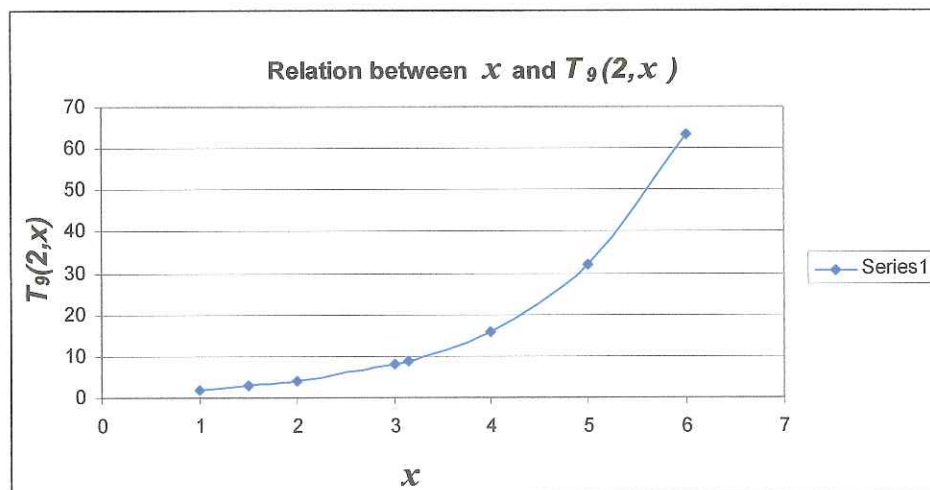
From this, one can see that the graph is exponential. Each sum T_n that is being approached for different values of a is the same as the value of a^x. From the graph one observes that as the value of x increases so does the value of $T_9(2, x)$. Larger values (such as 20) for a were also tested, however these were not included in the plot because more than 9 terms were needed in order to see what value of T_n was being approached.

I then considered the following sequence

$$t_0 = 1, \ t_1 = \frac{(x \ln 3)}{1}, \ t_2 = \frac{(x \ln 3)^2}{2 \times 1}, \ t_3 = \frac{(x \ln 3)^3}{3 \times 2 \times 1} \ \ldots, \ t_9 = \frac{(x \ln 3)^9}{9!}$$
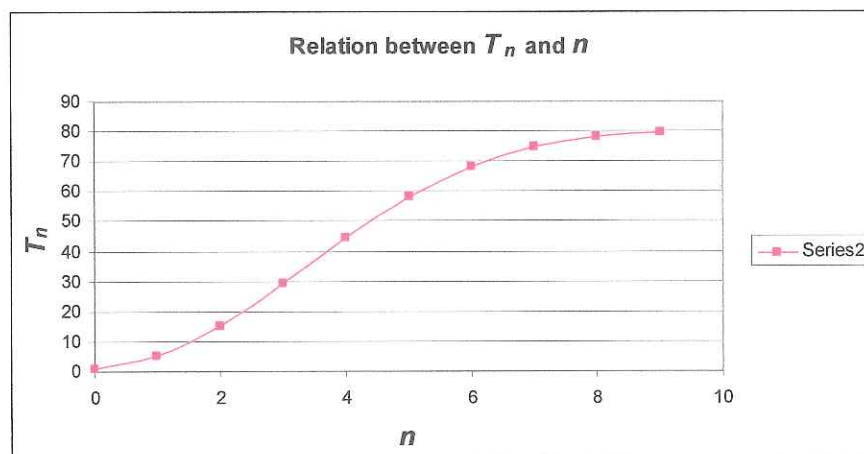
As $a = 3$, I calculated $T_9(3, x)$ for various positive values of x.

So, when $a = 3$ and $x = 2$ we get the following results
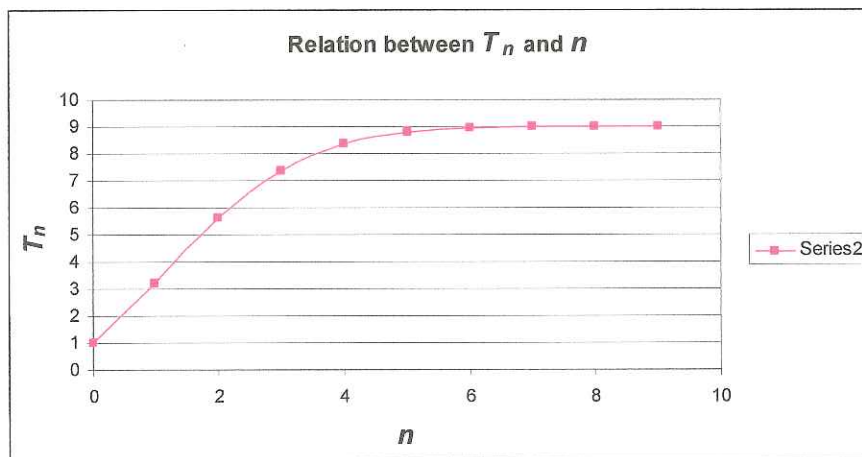
From this plot, one sees that as $n \to \infty$, then the value of $T_n \to 9$. This is the same as 3^2.

Next, I considered when $a = 3$ and $x = 4$. The results are represented on the following graph.

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From this plot, one cannot see exactly what value the sum T_9 is approaching. However from the previous test one can assume that as $n \to \infty$, then the value of $T_n \to 81$ because $3^4 = 81$. Also if one finds the value for more terms than just 9 terms, it is more clear that $T_n \to 81$.

More values for x were tested, then the results were presented on a table and a graph shown below. These show the relation between $T_9(3, x)$ and x.

x	$T_9(3, x)$
1.5	5.196105
2	8.999101
3	26.94128
3.141593	31.44939
4	79.80329

The above table is presented in the graph below.
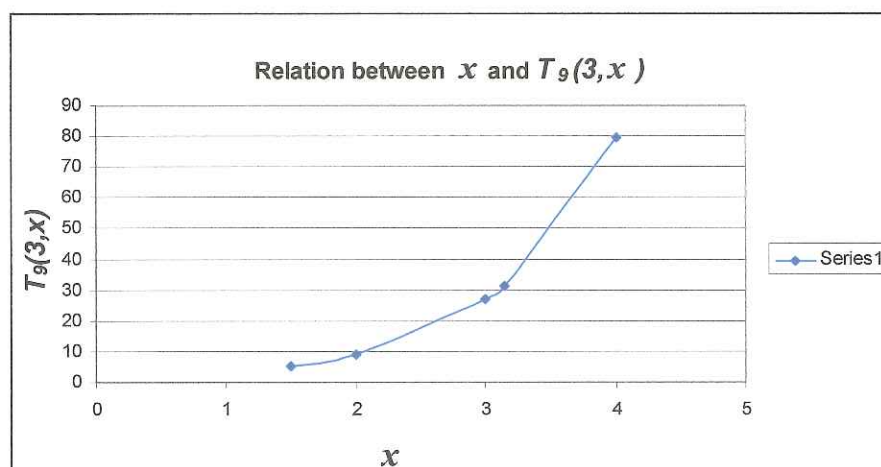

Relation between x and $T_9(3, x)$

One can see that the graph is exponential. Each sum T_n that is being approached for different values of a is the same as the value of a^x. From the graph one observes that as the value of x increases so does the value of $T_9(3, x)$. Greater numbers than $a = 4$ were tested, however 9 terms were not enough in order to see what these values of a were approaching (T_n).

I then continued this analysis to find the general statement for $T_n(a, x)$ as $n \to \infty$. I did this by first substituting larger natural numbers than I had already used. However in order to see what T_n approaches I had to go up to more than 9 terms.

So when $a = 2$ and $x = 20$, I got the following results.

n	T_n
0	1
2	110.9535
4	2093.887
6	16218.89
8	69574.03
10	193928.1
12	390151.6
14	613375
16	804952.6
18	933329.7
20	1002330
22	1032728
24	1043902
26	1047380
28	1048308
30	1048523
32	1048567
34	1048575
36	1048576

The above table is presented on the graph below.
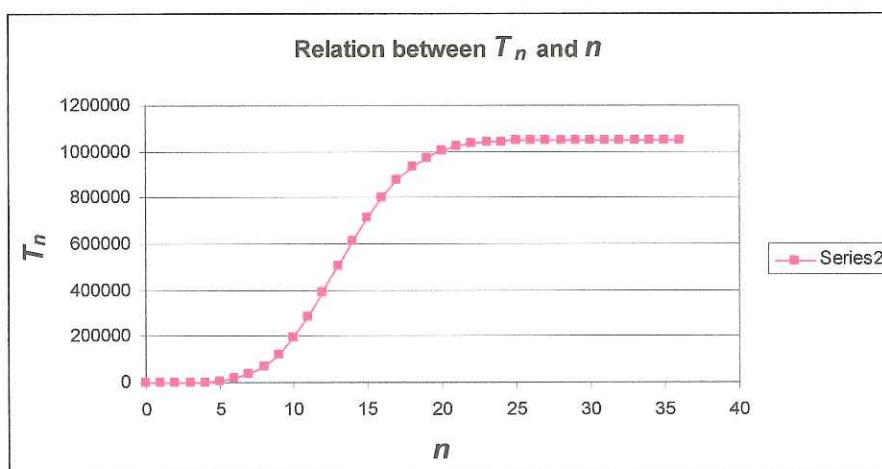
From this plot, one can see that T_n is approaching an asymptote. As seen from the table one needs to calculate T_n up to $n = 36$ in order to see what it is approaching. When $n \to \infty$, the value of $T_n \to 1048576$ (this value is the same as 2^{20}).

Thus my general statement for $T_n(a, x)$ as $n \to \infty$, is that $T_n \to a^x$. $x = 20$ agrees with my general statement, however as can be seen from the table one needs to calculate T_n for more than $n = 9$ for some sequences in order to be able to see what T_n

is approaching to. I then did some further testing in order to find the limitations of this general statement.

I tested $a = 2$ and $x = -2$, the results are shown in the graph below.

Relation between T_n and n
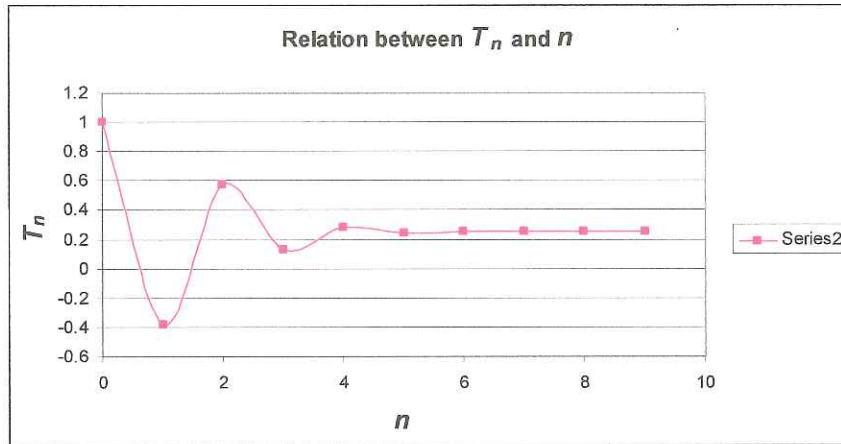
From the graph I concluded that when $n \to \infty$, the value of $T_n \to 0.25$. This value for T_n is the same as 2^{-2} and thus the same as a^x.

I also considered if $a = 3$ and $x = -4$. The results are shown in the graph below.

Relation between T_n and n
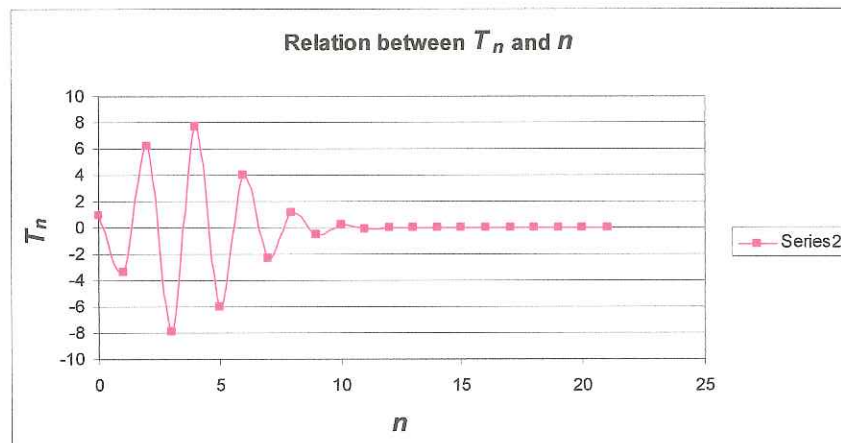
From the plot one can conclude that T_n is approaching an asymptote. When $n \to \infty$, the value of $T_n \to 0.012345$ which is the same as 3^{-4}.

Then I tested $x = 0$ and $a = 3$, the results are shown in the graph below.

Relation between T_n and n
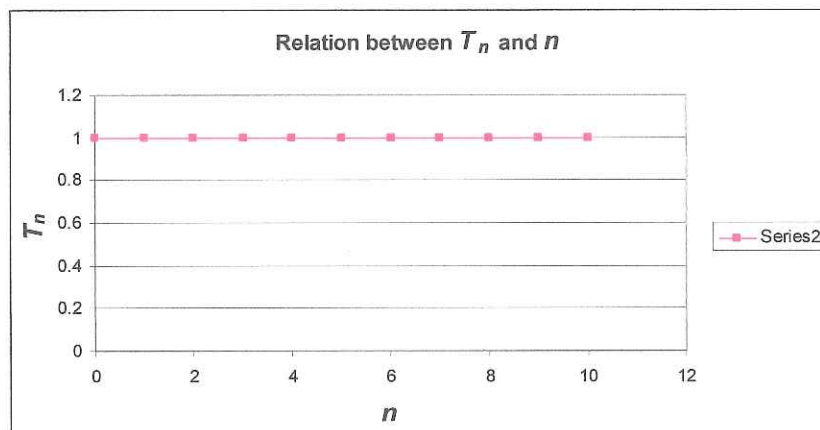
This plot shows that as $n \to \infty$, the value of sum $T_n = 1$, this value is the same as 3^0.

The investigation of whether $x \leq 0$ has helped me conclude that my general statement $T_n \to a^x$ is also relevant when $x \leq 0$. For $x = 0$, T_n does not approach an asymptote. Instead $T_n = 1$, however this still follows the general statement and therefore is part of the scope of the statement.

I then tested for different values of a.

Considering when a=4 and x=2, the results are on the table and graph below.

n	T_n
0	1
1	3.772589
2	7.616213
3	11.16848
4	13.63072
5	14.99607
6	15.627
7	15.8769
8	15.96351
9	15.99019
10	15.99759
11	15.99946
12	15.99989
13	15.99998
14	16
15	16

The above table is presented on the graph below.

Relation between T_n and n
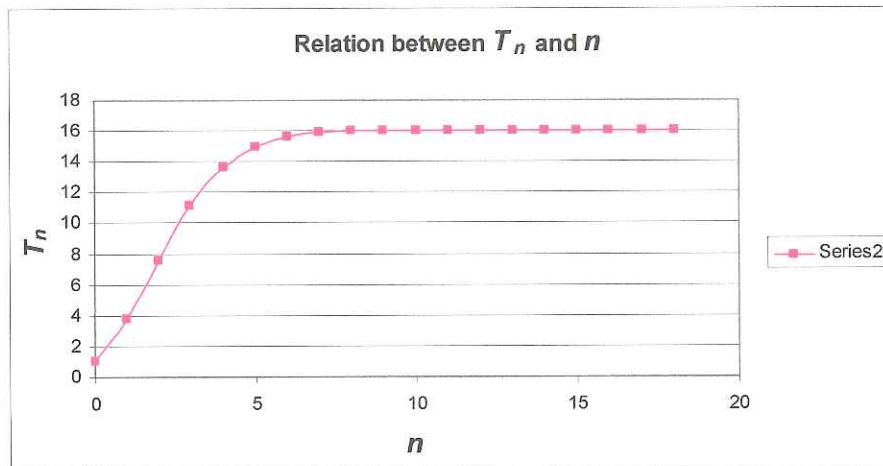
This plot shows that as $n \to \infty$, the value of $T_n \to 16$. This value is the same as 4^2 and thus a^x.

This investigation of a different value of a illustrates that $a > 0$. This is because $\ln 0$ or of any negative number would make the value of the term undefined. I know this because for example $\ln -2$ would be the same as saying e to the power of a number would equal -2, which is not possible.

Thus, from all the calculations and observations above I have concluded that for $T_n(a,x)$, as $n \to \infty$, $T_n \to a^x$. This statement is true for all positive values of a. *and any real x.* ✓