

**Quiz 1 (take home)**

**Ch. 1**

**Mathematics HL**

**Mr. Howes**

**DUE: Monday, September 3<sup>rd</sup>.**

**Name:.....**

**Please show all of your work on SEPARATE PAPER! Please be sure to include all appropriate steps for your proof by mathematical induction. You can see examples in the text, or see the examples I have posted on the website. It is OK to discuss problems together, but the work done must be your own. Of course, if you have questions, you can always ask me as well!**

- 1.** A geometric sequence  $u_1, u_2, u_3, \dots$  has  $u_1 = 27$  and a sum to infinity of  $\frac{81}{2}$ .

- (a) Find the common ratio of the geometric sequence.

**(2)**

An arithmetic sequence  $v_1, v_2, v_3, \dots$  is such that  $v_2 = u_2$  and  $v_4 = u_4$ .

- (b) Find the greatest value of  $N$  such that  $\sum_{n=1}^N v_n > 0$ .

**(5)**

**(Total 7 marks)**

- 2.** Prove by mathematical induction that, for  $n \in \mathbb{Z}^+$ ,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \dots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

**(Total 8 marks)**

3. Consider the arithmetic sequence 8, 26, 44, ....
- (a) Find an expression for the  $n^{\text{th}}$  term. (1)
- (b) Write down the sum of the first  $n$  terms using sigma notation. (1)
- (c) Calculate the sum of the first 15 terms. (2)  
(Total 4 marks)
4. A sum of \$ 5000 is invested at a compound interest rate of 6.3 % per annum.
- (a) Write down an expression for the value of the investment after  $n$  full years. (1)
- (b) What will be the value of the investment at the end of five years? (1)
- (c) The value of the investment will exceed \$10 000 after  $n$  full years.  
(i) Write an inequality to represent this information.  
(ii) Calculate the minimum value of  $n$ . (4)  
(Total 6 marks)

$$\textcircled{1} \text{ a) } S_{\infty} = \frac{U_1}{1-r}$$

KBY:

$$S_{\infty} = \frac{27}{1-r} = \frac{81}{2}$$

$$54 = 81(1-r)$$

$$54 = 81 - 81r$$

$$-27 = -81r$$

$$\frac{-27}{-81} = r$$

$$r = \frac{1}{3}$$

$$\text{b) } U_n = U_1 \cdot r^{n-1}$$

$$U_2 = 27 \left(\frac{1}{3}\right)^1 = 9 = V_2$$

$$U_4 = 27 \left(\frac{1}{3}\right)^3 = \frac{27}{27} = 1 = V_4$$

~~$\frac{-4}{V_1 - 4}$~~

$$\therefore V_1, 9, V_3, 1, V_5, \dots$$

$$U_n = U_1 + (n-1)d$$

$$\therefore d = -4.$$

$$\text{an } U_2 = 9 = V_1 + (2-1)(-4)$$

$$9 = V_1 - 4$$

$$\therefore V_1 = 13$$

We have 13, 9, 5, 1, -3, -7, -11, ... etc.

$\sum_{n=1}^N V_n > 0$  means what is the most ( $N$ ) amount of terms I can add up before the sum goes negative.

M1 test  $S_1 = 13$

$$S_2 = 13 + 9$$

$$S_3 = 13 + 9 + 5$$

$$S_4 = 13 + 9 + 5 + 1$$

$$S_7 = 13 + 9 + \dots + -11 = 7$$

$$S_8 = 13 + 9 + \dots + -15 = -8$$

$\therefore N = 7$  is the bigest.

$N$ .



$$\text{OR} \quad S_n = \frac{n}{2}(2u_1 + (n-1)d) > 0$$

$$\frac{n}{2}(2 \cdot 13 + (n-1) \cdot 4) > 0$$

$$\frac{n}{2}(26 - 4n + 4) > 0$$

$$\frac{n}{2}(30 - 4n) > 0$$

$$\therefore \frac{n}{2} > 0 \quad \text{or} \quad 30 - 4n > 0$$

$$n > 0 \quad \text{or} \quad 30 > 4n$$

$$7.5 > n$$

If  $n < 7.5$  then the biggest  $N$  is 7.

$$\textcircled{2} \quad \text{Prove} \quad 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \dots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$$

Proof. Let  $P(n)$  = the statement above.

Then LHS:  $S_n = 1$

$$\text{RHS: } 4 - \frac{1+2}{2^{1-1}} = 4 - \frac{3}{1} = 4 - 3 = 1$$

$\therefore P(1)$  is true.

Assume  $P(k)$ . That is  $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \dots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$ .

$$\text{Then } P(k+1) = \underbrace{1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \dots + k\left(\frac{1}{2}\right)^{k-1}}_{P(k)} + (k+1)\left(\frac{1}{2}\right)^{k-1} + U_{k+1}$$

$$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^{k-1}$$

$$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$$



2<sup>coat</sup>:

$$= 4 - \frac{2^k(k+2)}{2^k} + (k+1)\left(\frac{1^k}{2^k}\right)$$

$$= 4 - \frac{(2k+4)}{2^k} + \frac{(k+1)}{2^k}$$

$$= 4 + \frac{-2k-4}{2^k} + \frac{k+1}{2^k}$$

$$= 4 + \frac{-k-3}{2^k}$$

$$= 4 - \frac{(k+3)}{2^k}$$

$$= 4 - \frac{((k+1)+2)}{2^{(k+1)-1}} \Rightarrow P(k+1)$$

∴ If it is true that  $P(1)$  is true,  $P(k)$  is true, then  $P(k+1)$  is true also. Hence, by the principle of mathematical induction,  $P(n)$  is true for all  $n \in \mathbb{Z}^+$ .

③  $8, 26, 44, \dots$   $\therefore$  arithmetic

a) Now  $a_n = a_1 + (n-1)d$

$$a_n = 8 + (n-1)18$$

$$a_n = 8 + 18n - 18$$

$$a_n = 18n - 10$$

b)  $\sum_{r=1}^n (18n - 10)$

$$\begin{aligned} d) S_{15} &= \frac{15}{2} (2 \cdot 8 + (n-1)(18)) \\ &= \frac{15}{2} (16 + 18n - 18) \\ &= \frac{15}{2} (18n - 2) \\ &= \frac{15}{2} (18 \cdot 15 - 2) \\ &= \frac{15}{2} (268) \\ &= 2010. \end{aligned}$$

④ Let  $U_n = U_1 \cdot r^{n-1}$  represent the compound growth.

But... say \$100 @ 2% compounded yearly.

100	$100(1.02)$	$100(1.02)^2$	$100(1.02)^3$	$\dots$
$U_1$	$U_2$	$U_3$	$U_4$	
0 years	1 year	2 years	3 years	

$$\therefore \text{Use } \underbrace{U_{n+1}}_{\substack{\text{final} \\ \text{investment}}} = U_1(r)^n \xrightarrow{\substack{\text{initial} \\ \text{investment}}} \text{years.} \quad \xrightarrow{\text{rate}}$$

$$\therefore \text{a) } A = 5,000(100\% + 6.3\%)^n$$

$$A = 5,000(1.063)^n$$

$$\text{b) } A = 5,000(1.063)^5$$

$$A = 5,000(1.063)^5 \approx \$6786.35 \quad (\text{2 D.P.})$$

$$= \$6786 \quad (\text{nearest \$})$$

$$= \$6790 \quad (\text{3 SF})$$

$$\text{c) } \underbrace{10,000}_{\boxed{M1} \text{ GDC}} < \underbrace{5,000(1.063)^n}_{Y_2}$$

$\boxed{2^{nd}}$   $\boxed{\text{trace}}$  intersect.

$$\boxed{M2} \text{ logs! } 10,000 = 5,000(1.063)^n$$

$$\log(2) = \log(1.063^n)$$

$$\log(2) = n \cdot \log(1.063)$$

$$n = \frac{\log 2}{\log(1.063)} \approx 11.35$$

$\therefore$  More than 11 years or 12 full years to exceed \$10,000.