

- 46 $P(x)$ divided by $(x-1)(x-2)$ gives a remainder of $2x+3$.
 $\therefore P(x) = (x-1)(x-2)Q(x) + 2x+3$
for some polynomial $Q(x)$.
 $\therefore P(1) = (0)(-1)Q(1) + 2(1) + 3$
 $= 5$
 \therefore the remainder when $P(x)$ is divided by $x-1$ is 5.
{Remainder theorem}

- 47 Let $P(x) = x^3 - x^2 + (m+1)x + (2-m^2)$
Now $P(2) = 0$
 $\therefore 2^3 - 2^2 + (m+1)2 + (2-m^2) = 0$
 $\therefore 8 - 4 + 2m + 2 + 2 - m^2 = 0$
 $\therefore m^2 - 2m - 8 = 0$
 $\therefore (m-4)(m+2) = 0$
 $\therefore m = -2 \text{ or } 4$

If $m = -2$, $P(x) = x^3 - x^2 - x - 2$
 $= (x-2)(x^2 + ax + 1)$ for some a .

Equating coefficients of x^2 gives

$$-1 = -2 + a \\ \therefore a = 1$$

$$\therefore P(x) = (x-2)(x^2 + x + 1)$$

$$x^2 + x + 1 \text{ has } \Delta = 1^2 - 4 \times 1 \times 1 \\ = -3,$$

So no more real zeros exist.

$$\text{If } m = 4, \quad P(x) = x^3 - x^2 + 5x - 14 \\ = (x-2)(x^2 + ax + 7) \text{ for some } a.$$

Equating coefficients of x^2 gives

$$-1 = -2 + a \\ \therefore a = 1$$

$$\therefore P(x) = (x-2)(x^2 + x + 7)$$

$$x^2 + x + 7 \text{ has } \Delta = 1^2 - 4 \times 1 \times 7 \\ = -27,$$

so no more real zeros exist.

- 48 a Since a is a solution of the equation,

$$3a^3 - 11a^2 + 8a = 12a \\ \therefore 3a^3 - 11a^2 - 4a = 0 \\ \therefore a(3a^2 - 11a - 4) = 0 \\ \therefore a(3a+1)(a-4) = 0 \\ \therefore a = 0, -\frac{1}{3} \text{ or } 4$$

- b If $a = 0$, $3x^3 - 11x^2 + 8x = 0$

$$\therefore x(3x^2 - 11x + 8) = 0 \\ \therefore x(3x-8)(x-1) = 0 \\ \therefore x = 0, \frac{8}{3} \text{ or } 1$$

$$\text{If } a = -\frac{1}{3}, \quad 3x^3 - 11x^2 + 8x = 12(-\frac{1}{3}) \\ \therefore 3x^3 - 11x^2 + 8x + 4 = 0$$

$x = a$ is a solution, and so $(3x+1)$ must be a factor.

$$\therefore 3x^3 - 11x^2 + 8x + 4 = (3x+1)(x^2 + ax + 4)$$

$$\text{for some } a.$$

Equating coefficients of x^2 gives $-11 = 1 + 3a$
 $\therefore a = -4$

$$\therefore (3x+1)(x^2 - 4x + 4) = 0 \\ \therefore (3x+1)(x-2)^2 = 0 \\ \therefore x = -\frac{1}{3} \text{ or } 2$$

$$\text{If } a = 4, \quad 3x^3 - 11x^2 + 8x = 12(4) \\ \therefore 3x^3 - 11x^2 + 8x - 48 = 0$$

$x = a$ is a solution, so $(x-4)$ is a factor.

$$\therefore 3x^3 - 11x^2 + 8x - 48 = (x-4)(3x^2 + ax + 12)$$

$$\text{for some } a.$$

Equating coefficients of x^2 gives $-11 = a - 12$
 $\therefore a = 1$

$$\therefore (x-4)(3x^2 + x + 12) = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 3 \times 12}}{2 \times 3}$$

$$x = 4 \text{ or } \frac{-1 \pm i\sqrt{143}}{6}$$

- 49 $P(1) = 6$ {Remainder theorem}

$$\therefore 2 + 3 + p = 6 \\ \therefore p = 1$$

- $P(2) = 77$ {Remainder theorem}

$$\therefore 2^{m+1} + 3 \times 2^n + 1 = 77 \\ \therefore 2^{m+1} = 76 - 3 \times 2^n$$

We need to find $n \in \mathbb{Z}^+$ such that $76 - 3 \times 2^n$ is a power of 2 which is $\geq 2^2$. $\{n \in \mathbb{Z}^+\}$

| n | $76 - 3 \times 2^n$ |
|-----|---------------------|
| 1 | 70 |
| 2 | 64 |
| 3 | 52 |
| 4 | 28 |
| 5 | -20 |

✓

too small

$$\therefore n = 2, \quad 2^{m+1} = 64 \\ \therefore m = 5$$

So, the only solution is $m = 5, n = 2, p = 1$.

- 50 $P(i) = 0 \quad \therefore 6i^4 + 7i^3 + 8i^2 + 7i + k = 0$

$$\therefore 6 - 7i - 8 + 7i + k = 0 \\ \therefore k = 2$$

Since $P(z)$ is a real polynomial, $-i$ must also be a zero.

$$\therefore (z+i)(z-i) = z^2 + 1 \text{ is a factor of } P(z)$$

$$\therefore P(z) = (z^2 + 1)(6z^2 + az + 2) \text{ for some } a$$

Equating coefficients of x^3 , $a = 7$

$$\therefore P(z) = (z^2 + 1)(6z^2 + 7z + 2) \\ = (z^2 + 1)(3z + 2)(2z + 1)$$

∴ the zeros of $P(z)$ are $\pm i, -\frac{2}{3}$ and $-\frac{1}{2}$

- 51 $P(x)$ is a real polynomial, so $3+2i$ must also be a zero of $P(x)$.

$$(3+2i) + (3-2i) = 6 \text{ and}$$

$$(3+2i)(3-2i) = 9+4 \\ = 13$$

So, $x^2 - 6x + 13$ is a factor of $P(x)$.

$$\therefore 2x^3 + mx^2 - (m+1)x + (3-4m)$$

$$= (x^2 - 6x + 13)(2x+b) \text{ for some } b$$

$$= 2x^3 + (b-12)x^2 + (26-6b)x + 13b$$

Equating coefficients of x^2 : $m = b-12 \dots (1)$

Equating constants: $3-4m = 13b \dots (2)$

Substituting (1) into (2): $13b = 3 - 4(b-12)$

$$\therefore 13b = 3 - 4b + 48$$

$$\therefore 17b = 51$$

$$\therefore b = 3$$

$$\therefore m = 3 - 12 = -9$$

$$\therefore P(x) = (x^2 - 6x + 13)(2x+3)$$

∴ the zeros of $P(x)$ are $3 \pm 2i$ and $-\frac{3}{2}$

$$52 \quad a^2 \times a^3 + a^2 - a^4 \times a - 2 = 0$$

$$\therefore a^2 - 2 = 0$$

$$\therefore a = \pm\sqrt{2}$$

$$\therefore P(z) = 2z^3 + z^2 - 4z - 2$$

Since $P(z)$ is the same whether $a = \pm\sqrt{2}$, both $z = \sqrt{2}$ and $z = -\sqrt{2}$ must be zeros of $P(z)$.

$$\text{Hence } (z - \sqrt{2})(z + \sqrt{2}) = (z^2 - 2) \text{ is a factor of } P(z).$$

$$\therefore P(z) = (z^2 - 2)(2z + 1)$$

∴ the zeros of $P(z)$ are $\pm\sqrt{2}$ and $-\frac{1}{2}$

- 53 $(x+1)$ and $(x-2)^2$ must be factors, given the x -intercepts of $P(x)$.

$$\text{Also, } P(0) = 56 \text{ and } P(1) = 20.$$

$$\text{So } P(x) = (ax+b)(x+1)(x-2)^2$$

$$P(0) = 56$$

$$\therefore (a \times 0 + b)(1)(-2)^2 = 56$$

$$\therefore 4b = 56$$

$$\therefore b = 14$$

and $P(1) = 20$

$$\therefore (a \times 1 + 14)(2)(-1)^2 = 20$$

$$(a+14) \times 2 = 20$$

$$\therefore a = -4$$

$$\therefore P(x) = (-4x+14)(x+1)(x-2)^2$$

$$= (-4x+14)(x+1)(x^2 - 4x + 4)$$

$$= (-4x+14)(x^3 - 3x^2 + 4)$$

$$= -4x^4 + 26x^3 - 42x^2 - 16x + 56$$

- 54 Let $P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$

Since bi is a zero of the real polynomial $P(x)$, so is $-bi$.

∴ $x^2 + b^2$ is a factor of $P(x)$

$$\therefore P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$$

$$= (x^2 + b^2)(x^2 + cx + d) \text{ for some } c, d$$

$$= x^4 + cx^3 + (b^2 + d)x^2 + b^2cx + b^2d$$

Equating the coefficients of x^3 : $c = 2$

Equating the coefficients of x : $2b^2 = 6$

$$\therefore b = \pm\sqrt{3}$$

Equating the coefficients of x^2 : $3+d = 8$

$$\therefore d = 5$$

$$\therefore P(x) = (x^2 + 3)(x^2 + 2x + 5)$$

$$\text{Now } x^2 + 2x + 5 = 0$$

$$\text{when } x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= -1 \pm 2i$$

∴ the zeros are $\pm\sqrt{3}i, -1 \pm 2i$

SOLUTIONS TO TOPIC 3 (CIRCULAR FUNCTIONS AND TRIGONOMETRY)

1 a $\frac{2\pi}{9}$ radians

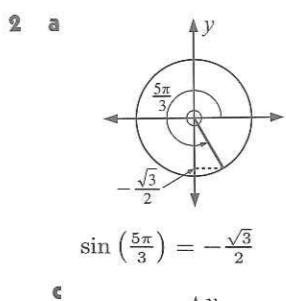
$$= (\frac{2\pi}{9} \times \frac{180}{\pi})^\circ$$

$$= 40^\circ$$

b 140°

$$= (140 \times \frac{\pi}{180}) \text{ radians}$$

$$= \frac{7\pi}{9}$$



$$\sin(\frac{5\pi}{3}) = -\frac{\sqrt{3}}{2}$$

8 a

$$\frac{\sin 2\theta}{8} = \frac{\sin \theta}{5} \quad \{ \text{sine rule} \}$$

$$\therefore \frac{2 \sin \theta \cos \theta}{8} = \frac{\sin \theta}{5}$$

$$\therefore \cos \theta = \frac{4}{5}$$

b

$$\widehat{ABC} = \pi - 3\theta$$

$$= \pi - 3 \arccos\left(\frac{4}{5}\right)$$

$$\approx 1.211$$

$$\text{Area of triangle} \approx \frac{1}{2} \times 5 \times 8 \times \sin(1.211)$$

$$\approx 18.7 \text{ cm}^2$$

9

$$11^2 = x^2 + 12^2 - 2 \times x \times 12 \cos 60^\circ$$

$$\therefore 121 = x^2 + 144 - 12x$$

$$\therefore x^2 - 12x + 23 = 0$$

$$\therefore x = \frac{12 \pm \sqrt{(-12)^2 - 4 \times 1 \times 23}}{2 \times 1}$$

$$\therefore x = 6 \pm \sqrt{13}, \text{ so } PQ = 6 \pm \sqrt{13} \text{ cm}$$

- 10 a A vertical stretch with factor 2, and a horizontal stretch with factor 3.
- b A translation of $\frac{\pi}{3}$ units to the left, and a translation of 4 units downwards.

11 a Period = $\frac{2\pi}{3}$

b Period = $\frac{2\pi}{\frac{1}{2}} = 4\pi$

c

Period = π

$$\text{or } \sin^2 x + 5 = \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) + 5$$

$$= -\frac{1}{2} \cos 2x + \frac{\pi}{2}$$

$$\therefore \text{period} = \frac{2\pi}{2} = \pi$$

- 12 For the sine function $y = A \sin B(x - C) + D$:
The amplitude = 2, so $A = 2$.

The period = π , so $\frac{2\pi}{B} = \pi \Rightarrow B = 2$.

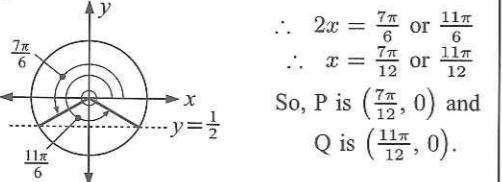
The principal axis is $y = 1$, so $D = 1$.

There is no horizontal translation, so $C = 0$.

∴ the function is $y = 2 \sin(2x) + 1$

We want to solve $2 \sin(2x) + 1 = 0$, $0 \leq x \leq \pi$

$$\therefore \sin 2x = -\frac{1}{2}$$



$$\therefore 2x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

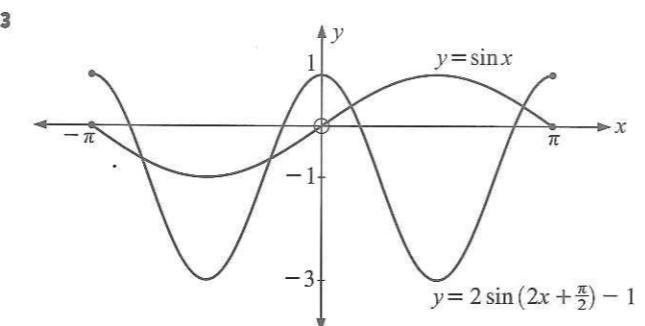
$$\therefore x = \frac{7\pi}{12} \text{ or } \frac{11\pi}{12}$$

$$\text{So, P is } (\frac{7\pi}{12}, 0) \text{ and}$$

$$\text{Q is } (\frac{11\pi}{12}, 0).$$

$$\frac{\sin 2\theta}{8} = \frac{\sin \theta}{5} \quad \{ \text{sine rule} \}$$

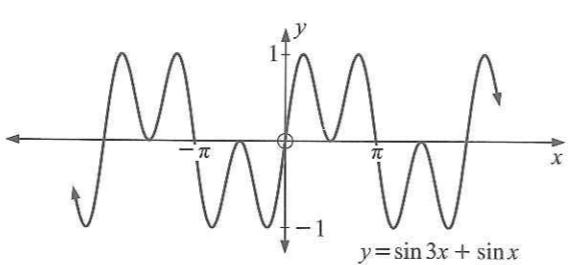
$$\therefore \cos \theta = \frac{4}{5}$$



- 14 a Amplitude = 1
The principal axis is $y = 0$
Period = $\frac{2\pi}{4} = \frac{\pi}{2}$
- b Amplitude = 2
The principal axis is $y = -1$
Period = $\frac{2\pi}{\frac{1}{2}} = 4\pi$

15 a Period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$

b Period = $\frac{\pi}{5}$



$y = \sin 3x$ has period $\frac{2\pi}{3}$, and $y = \sin x$ has period 2π .
So, $y = \sin 3x + \sin x$ has period 2π .

16 $\tan 2x$ has period $\frac{\pi}{2}$, and $\tan 3x$ has period $\frac{\pi}{3}$.
The lowest common multiple of $\frac{\pi}{2}$ and $\frac{\pi}{3}$ is π .
∴ period = π

- 17 a $\csc(x) = \frac{1}{\sin x}$
∴ vertical asymptotes occur when $\sin x = 0$
∴ the vertical asymptotes are $x = 0, \pm\pi$ and $\pm 2\pi$
- b $\sec(2x) = \frac{1}{\cos 2x}$
∴ vertical asymptotes occur when $\cos 2x = 0$

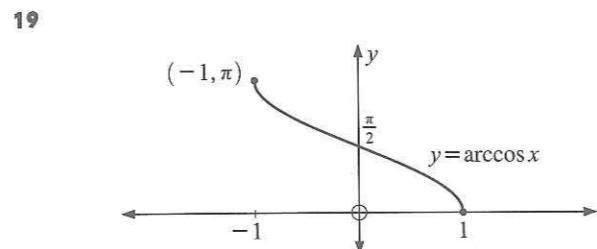
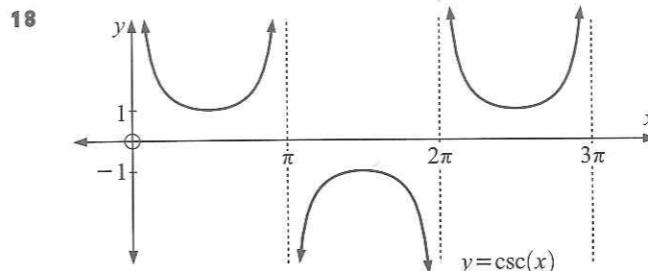
$$\therefore 2x = \pm\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$

$$\therefore x = \pm\frac{\pi}{4} + k\pi, k \in \mathbb{Z}$$

$$\therefore \text{the vertical asymptotes are}$$

$$x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4} \text{ and } \pm\frac{7\pi}{4}$$

- c $\cot\left(\frac{x}{2}\right) = \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$
∴ vertical asymptotes occur when $\sin\left(\frac{x}{2}\right) = 0$
- $$\therefore \frac{x}{2} = 0 + k\pi, k \in \mathbb{Z}$$
- $$\therefore x = 2k\pi, k \in \mathbb{Z}$$
- $$\therefore x = 0 \text{ and } x = \pm 2\pi$$
- $$\therefore \text{the vertical asymptotes are } x = 0 \text{ and } x = \pm 2\pi.$$



20 $\arcsin(-\frac{1}{2}) + \arctan(1) + \arccos(-\frac{1}{2})$

$$= -\frac{\pi}{6} + \frac{\pi}{4} + \frac{2\pi}{3}$$

$$= \frac{3\pi}{4}$$

21 a $\sin(\arccos(-\frac{\sqrt{3}}{2}))$

$$= \sin(\frac{5\pi}{6})$$

$$= \frac{1}{2}$$

b $\tan(\arcsin \frac{1}{\sqrt{2}})$

$$= \tan(\frac{\pi}{4})$$

$$= 1$$

22 $\sin(\frac{3\pi}{2} - \phi) \tan(\phi + \pi)$

$$= (\sin \frac{3\pi}{2} \cos \phi - \cos \frac{3\pi}{2} \sin \phi) \tan \phi$$

$$= ((-1) \cos \phi - 0 \times \sin \phi) \frac{\sin \phi}{\cos \phi}$$

$$= -\sin \phi$$

23 $1 - \frac{\sin^2 \theta}{1 + \cos \theta} = \frac{1 + \cos \theta}{1 + \cos \theta} - \frac{\sin^2 \theta}{1 + \cos \theta}$

$$= \frac{1 + \cos \theta - (1 - \cos^2 \theta)}{1 + \cos \theta}$$

$$= \frac{\cos \theta + \cos^2 \theta}{1 + \cos \theta}$$

$$= \frac{\cos \theta(1 + \cos \theta)}{1 + \cos \theta}$$

$$= \cos \theta$$

24 $\cos 79^\circ \cos 71^\circ - \sin 79^\circ \sin 71^\circ = \cos(79^\circ + 71^\circ)$

$$= \cos(150^\circ)$$

$$= -\frac{\sqrt{3}}{2}$$

25 $\cos 2x = \frac{5}{8}$

∴ $1 - 2 \sin^2 x = \frac{5}{8} \quad \{ \text{double angle formula} \}$

∴ $2 \sin^2 x = \frac{3}{8}$

∴ $\sin x = \pm \frac{\sqrt{3}}{4}$

26 $\sin^2 \theta + \cos^2 \theta = 1$

∴ $(\frac{2}{3})^2 + \cos^2 \theta = 1$

∴ $\cos^2 \theta = \frac{5}{9}$

∴ $\cos \theta = -\frac{\sqrt{5}}{3} \quad \{ \theta \text{ is obtuse} \}$

$\sin 2\theta = 2 \sin \theta \cos \theta$

$$= 2 \times \frac{2}{3} \times -\frac{\sqrt{5}}{3}$$

$$= -\frac{4\sqrt{5}}{9}$$

27 $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

$$= \frac{2 \times 2}{1 - 2^2}$$

$$= -\frac{4}{3}$$

$$= \frac{-\frac{4}{3} + 2}{1 - (-\frac{4}{3}) \times 2}$$

$$= \frac{2}{11}$$

28

$\tan 2A = \sin A$

∴ $\frac{\sin 2A}{\cos 2A} = \sin A$

∴ $2 \sin A \cos A = \sin A \quad \{ \text{double angle formula} \}$

∴ $2 \cos^2 A - 1 = \sin A$

∴ $\frac{2 \cos A}{2 \cos^2 A - 1} = 1 \quad \{ \sin A \neq 0 \}$

$2 \cos A = 2 \cos^2 A - 1$

∴ $2 \cos^2 A - 2 \cos A - 1 = 0$

∴ $\cos A = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2 \times (-1)}}{2 \times 2}$

$$= \frac{1 \pm \sqrt{3}}{2}$$

But $|\cos A| \leq 1$, so $\cos A = \frac{1 - \sqrt{3}}{2}$

29 $\sin x - 2 \cos x = A \sin(x + \alpha)$

$$= A(\sin x \cos \alpha + \cos x \sin \alpha)$$

$$= A \sin x \cos \alpha + A \cos x \sin \alpha$$

Equating the coefficients of $\sin x$ and $\cos x$:

$A \cos \alpha = 1 \quad \text{and} \quad A \sin \alpha = -2$

∴ $\cos \alpha = \frac{1}{A} \quad \text{and} \quad \sin \alpha = -\frac{2}{A}$

Now $\sin^2 \alpha + \cos^2 \alpha = 1$

∴ $\left(\frac{-2}{A}\right)^2 + \left(\frac{1}{A}\right)^2 = 1$

$$\therefore \frac{4+1}{A^2} = 1$$

$$\therefore A^2 = 5$$

∴ $A = \sqrt{5} \quad \{ A > 0 \}$

So $\cos \alpha = \frac{1}{\sqrt{5}}$, $\sin \alpha = -\frac{2}{\sqrt{5}}$

∴ α is in the 4th quadrant.

∴ $\alpha \approx 5.18$

30 $\tan 2\theta = 2$

∴ $\frac{2 \tan \theta}{1 - \tan^2 \theta} = 2 \quad \{ \text{double angle formula} \}$

$2 \tan \theta = 2 - 2 \tan^2 \theta$

$\tan^2 \theta + \tan \theta - 1 = 0$

∴ $\tan \theta = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-1)}}{2 \times 1}$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

Now $2\theta \in [\pi, \frac{3\pi}{2}]$, so $\theta \in [\frac{\pi}{2}, \frac{3\pi}{4}]$

∴ $\tan \theta = \frac{-1 - \sqrt{5}}{2} \quad \{ \tan \theta < 0 \}$

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$$\begin{aligned} \sin x &= 2 \sin\left(x - \frac{\pi}{6}\right) \\ \therefore \sin x &= 2 \left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right) \\ &= 2 \sin x \left(\frac{\sqrt{3}}{2}\right) - 2 \cos x \left(\frac{1}{2}\right) \\ \therefore \sin x \left(1 - \sqrt{3}\right) &= -\cos x \\ \therefore \frac{\sin x}{\cos x} &= -\frac{1}{1 - \sqrt{3}} \\ \therefore \tan x &= \frac{1}{\sqrt{3} - 1} \end{aligned}$$

$$\begin{aligned} 32 \quad \csc(2x) - \cot(2x) &= \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} \\ &= \frac{1 - \cos 2x}{\sin 2x} \\ &= \frac{1 - (1 - 2 \sin^2 x)}{2 \sin x \cos x} \\ &\quad \text{[double angle formulae]} \\ &= \frac{2 \sin^2 x}{2 \sin x \cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \\ \therefore \tan\left(\frac{5\pi}{12}\right) &= \csc\left(\frac{5\pi}{6}\right) - \cot\left(\frac{5\pi}{6}\right) \\ &= \frac{1}{\sin\left(\frac{5\pi}{6}\right)} - \frac{1}{\tan\left(\frac{5\pi}{6}\right)} \\ &= \frac{1}{\frac{1}{2}} - \frac{1}{-\frac{1}{\sqrt{3}}} \\ &= 2 + \sqrt{3} \end{aligned}$$

$$\begin{aligned} 33 \quad \frac{1}{\tan \theta - \sec \theta} &= \frac{1}{\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta}} \quad \{\cos \theta \neq 0\} \\ &= \frac{1}{\left(\frac{\sin \theta - 1}{\cos \theta}\right)} \\ &= \frac{\cos \theta}{\sin \theta - 1} \times \left(\frac{\sin \theta + 1}{\sin \theta + 1}\right) \\ &= \frac{\cos \theta \sin \theta + \cos \theta}{\sin^2 \theta - 1} \\ &= \frac{\cos \theta \sin \theta + \cos \theta}{-\cos^2 \theta} \\ &= -\frac{\cos \theta \sin \theta}{\cos^2 \theta} - \frac{\cos \theta}{\cos^2 \theta} \\ &= -\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta} \\ &= -(\tan \theta + \sec \theta) \end{aligned}$$

$$\begin{aligned} 34 \quad \sin 2x &= \sin x, \quad x \in [-\pi, \pi] \\ \therefore 2 \sin x \cos x - \sin x &= 0 \\ \therefore \sin x(2 \cos x - 1) &= 0 \\ \therefore \sin x = 0 \quad \text{or} \quad \cos x = \frac{1}{2} \\ \therefore x = 0, \pm \frac{\pi}{3} \quad \text{or} \quad \pm \pi \\ \therefore x = 0, \pm \frac{\pi}{3} \quad \text{or} \quad \pm \pi \end{aligned}$$

$$\begin{aligned} 35 \quad 2 \sin^2 x - \cos x &= 1, \quad x \in [0, 2\pi] \\ \therefore 2(1 - \cos^2 x) - \cos x &= 1 \\ \therefore 2 - 2 \cos^2 x - \cos x &= 1 \\ \therefore 2 \cos^2 x + \cos x - 1 &= 0 \\ \therefore (2 \cos x - 1)(\cos x + 1) &= 0 \\ \cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1 & \end{aligned}$$

$$\begin{aligned} 36 \quad \sin x + \cos x &= 1, \quad 0 \leq x \leq \pi \\ \therefore \sin^2 x + 2 \sin x \cos x + \cos^2 x &= 1 \quad \dots (*) \\ &\quad \text{[squaring both sides]} \\ \therefore \sin 2x + 1 &= 1 \\ \therefore \sin 2x &= 0 \\ \therefore 2x = 0 + k\pi, \quad k \in \mathbb{Z} & \\ \therefore x = \frac{k\pi}{2}, \quad k \in \mathbb{Z} & \\ \therefore x = 0, \frac{\pi}{2}, \pi \quad \{0 \leq x \leq \pi\} & \end{aligned}$$

Since we squared both sides at (*), we need to check our solutions.

$$\begin{aligned} \sin 0 + \cos 0 &= 1 & \checkmark \\ \sin \frac{\pi}{2} + \cos \frac{\pi}{2} &= 1 & \checkmark \\ \sin \pi + \cos \pi &= -1 & \times \\ \text{So } x = 0 \text{ or } \frac{\pi}{2} & \end{aligned}$$

$$\begin{aligned} 37 \quad \arcsin(2x - 3) &= -\frac{\pi}{6} \quad 38 \quad \cos 2\alpha = \sin^2 \alpha \\ \therefore 2x - 3 &= \sin\left(-\frac{\pi}{6}\right) \\ \therefore 2x - 3 &= -\frac{1}{2} \\ \therefore 2x &= \frac{5}{2} \\ \therefore x &= \frac{5}{4} \\ \therefore 1 - 2 \sin^2 \alpha &= \sin^2 \alpha \\ \therefore 1 &= 3 \sin^2 \alpha \\ \therefore \sin^2 \alpha &= \frac{1}{3} \\ \therefore \cos^2 \alpha &= \frac{2}{3} \\ \therefore \cot^2 \alpha &= \frac{\cos^2 \alpha}{\sin^2 \alpha} = 2 \\ \therefore \cot \alpha &= \pm \sqrt{2} \end{aligned}$$

$$\begin{aligned} 39 \quad \cot \theta + \tan \theta &= 2, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \therefore \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} &= 2 \\ \therefore \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} &= 2 \\ \therefore 1 &= 2 \sin \theta \cos \theta \\ \therefore 1 &= \sin 2\theta \\ \therefore 2\theta &= \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z} \\ \therefore \theta &= \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z} \\ \therefore \theta &= \frac{\pi}{4}, \quad \{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\} \end{aligned}$$

$$\begin{aligned} 40 \quad \text{a} \quad \sqrt{3} \tan\left(\frac{x}{2}\right) &= -1, \quad x \in [-\pi, 3\pi] \\ \therefore \tan\left(\frac{x}{2}\right) &= -\frac{1}{\sqrt{3}} \\ \therefore \frac{x}{2} &= \frac{5\pi}{6} + k\pi, \quad k \in \mathbb{Z} \\ \therefore x &= \frac{5\pi}{3} + 2k\pi, \quad k \in \mathbb{Z} \\ \therefore x &= \frac{5\pi}{3} \text{ or } -\frac{\pi}{3} \quad \{x \in [-\pi, 3\pi]\} \end{aligned}$$

$$\begin{aligned} \text{b} \quad \sqrt{3} + 2 \sin(2x) &= 0, \quad x \in [-\pi, 3\pi] \\ \therefore \sin 2x &= -\frac{\sqrt{3}}{2} \\ \therefore 2x &= \left\{ \frac{\frac{4\pi}{3}}{3}, \frac{\frac{5\pi}{3}}{3} \right\} + k2\pi, \quad k \in \mathbb{Z} \\ \therefore x &= \left\{ \frac{\frac{2\pi}{3}}{3}, \frac{\frac{5\pi}{6}}{3} \right\} + k\pi, \quad k \in \mathbb{Z} \\ \therefore x &= -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}, \frac{8\pi}{3} \text{ or } \frac{17\pi}{6} \\ \{x \in [-\pi, 3\pi]\} & \end{aligned}$$

$$\begin{aligned} 41 \quad \sin x + \sqrt{3} \cos x &= 0 \\ \therefore \sin x &= -\sqrt{3} \cos x \\ \therefore \frac{\sin x}{\cos x} &= -\sqrt{3} \\ \therefore \tan x &= -\sqrt{3} \\ \therefore x &= \frac{2\pi}{3} + k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} 42 \quad \frac{\sin \theta + 2 \cos \theta}{\sin \theta - \cos \theta} &= 2 \\ \therefore \sin \theta + 2 \cos \theta &= 2(\sin \theta - \cos \theta) \\ \therefore 4 \cos \theta &= \sin \theta \\ \therefore \tan \theta &= 4 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ &= \frac{2 \times 4}{1 - 4^2} \\ &= -\frac{8}{15} \end{aligned}$$

SOLUTIONS TO TOPIC 4 (MATRICES)

1 To evaluate $2\mathbf{A} - 3\mathbf{B}$, \mathbf{A} and \mathbf{B} must have the same order. Since \mathbf{A} has order $m \times n$, \mathbf{B} must have order $m \times n$. So, $p = m$ and $q = n$.

2 Let the cost of 1 soccer ball be $\$x$, the cost of 1 softball be $\$y$ and the cost of 1 basketball be $\$z$.

School A paid $2x + 1y + 3z = 90$.

School B paid $3x + 2y + 1z = 81$.

School C paid $5x + 0y + 2z = 104$.

$$\text{In matrix form, } \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 5 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 90 \\ 81 \\ 104 \end{pmatrix}$$

This system has solution $x = 14$, $y = 11$, $z = 17$.

So, a soccer ball costs $\$14$, a softball costs $\$11$ and a basketball costs $\$17$.

- 3 $\mathbf{A} = \mathbf{B}$ if: (1) \mathbf{A} and \mathbf{B} have the same order
(2) each element a_{ij} of \mathbf{A} is equal to the corresponding element b_{ij} of \mathbf{B} .

4 a If \mathbf{A} has order $m \times n$ and \mathbf{B} has order $n \times p$ then \mathbf{C} has order $m \times p$ and $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$

$$\mathbf{b} \quad c_{34} = \sum_{k=1}^n a_{3k} b_{k4}$$

5 If \mathbf{O} is the matrix with all entries 0, then $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$ if \mathbf{A} and \mathbf{O} have the same order. $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$

6 If \mathbf{I} is $n \times n$ then \mathbf{A} must be of order $n \times n$. The elements, e_{ij} of \mathbf{I} are: $e_{ii} = 1$, for all i
 $e_{ij} = 0$, for all $i \neq j$.

7 $\begin{pmatrix} k & 5 \\ 1 & k \end{pmatrix}$ is singular if $\begin{vmatrix} k & 5 \\ 1 & k \end{vmatrix} = k^2 - 5 = 0$
if $k = \pm\sqrt{5}$. If $\begin{pmatrix} k & 5 \\ 1 & k \end{pmatrix}$ is singular it has no inverse.

8 Assuming \mathbf{A} and \mathbf{B} are square matrices with inverses \mathbf{A}^{-1} and \mathbf{B}^{-1} respectively, $3\mathbf{A}\mathbf{X}\mathbf{B} - 2\mathbf{C} = 0$

$$\therefore 3\mathbf{A}\mathbf{X}\mathbf{B} = 2\mathbf{C}$$

$$\therefore 3\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{B}^{-1} = 2\mathbf{C}\mathbf{B}^{-1}$$

$$\therefore 3\mathbf{AX} = 2\mathbf{CB}^{-1}$$

$$\therefore \mathbf{AX} = \frac{2}{3}\mathbf{CB}^{-1}$$

$$\therefore \mathbf{A}^{-1}\mathbf{AX} = \frac{2}{3}\mathbf{A}^{-1}\mathbf{CB}^{-1}$$

$$\therefore \mathbf{X} = \frac{2}{3}\mathbf{A}^{-1}\mathbf{CB}^{-1}$$

9 $\begin{pmatrix} 2 & -1 & 1 \\ m & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}$ has an inverse if its determinant $\neq 0$.

$$\begin{vmatrix} 2 & -1 & 1 \\ m & 2 & 1 \\ 3 & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} m & 1 \\ 3 & 2 \end{vmatrix} + 1 \begin{vmatrix} m & 2 \\ 3 & -1 \end{vmatrix} \\ = 2(5) + 1(2m - 3) + (-m - 6) \\ = 10 + 2m - 3 - m - 6 \\ = m + 1 \end{math>$$

So, the matrix has an inverse if $m \neq -1$.

10 The equations have a unique solution if $\begin{vmatrix} 3 & -a & 2 \\ 1 & 2 & -3 \\ -1 & -1 & 1 \end{vmatrix} \neq 0$.

$$\begin{vmatrix} 3 & -a & 2 \\ 1 & 2 & -3 \\ -1 & -1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} - (-a) \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} \\ = 3(-1) + a(-2) + 2(1) \\ = -1 - 2a \end{math>$$

which is 0 if $a = -\frac{1}{2}$.

Thus, we have a unique solution if $a \neq -\frac{1}{2}$.

$$\det \mathbf{A} = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 6 - -1 = 7 \quad \text{and}$$

$$\det \mathbf{B} = \begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} = -6 - 5 = -11$$

So, $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B} = 7 \times -11 = -77$

12 Writing the system in augmented form

$$\begin{array}{ccc|c} 1 & 3 & k & 2 \\ k & -2 & 3 & k \\ 4 & -3 & 10 & 5 \end{array} \sim \begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2 - 3k & 3 - k^2 & -k \\ 0 & -15 & 10 - 4k & -3 \end{array} \xrightarrow{R_2 \rightarrow R_2 - kR_1} \begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2 - 3k & 3 - k^2 & -k \\ 0 & -15 & 10 - 4k & -3 \end{array} \xrightarrow{R_3 \rightarrow R_3 - 4R_1} \begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2 - 3k & 3 - k^2 & -k \\ 0 & -2 - 3k & 3 - k^2 & -k \end{array} \sim \begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -15 & 10 - 4k & -3 \\ 0 & 0 & 25 - 22k - 3k^2 & 6 - 6k \end{array} \xrightarrow{R_3 \rightarrow 15R_3 - (2 + 3k)R_2} \begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -15 & 10 - 4k & -3 \\ 0 & 0 & 25 - 22k - 3k^2 & 6 - 6k \end{array}$$