

Type I tasks**1 Continued fractions****SL Type I****Description**

Consider the continued fraction below.

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\dots}}}}}}$$

We can consider this "infinite fraction" as a sequence of terms, t_n , where

$$t_1 = 1 + 1$$

$$t_2 = 1 + \cfrac{1}{1+1}$$

$$t_3 = 1 + \cfrac{1}{1 + \cfrac{1}{1+1}}$$

•

•

•

Method

1. Determine a generalized formula for t_{n+1} in terms of t_n .
2. Compute the decimal equivalents of the first 10 terms. Enter the terms into a data table and plot the relation between n and t_n using a GDC or computer. Provide printed output of your plot. What do you notice? What does this suggest about the value of $t_n - t_{n+1}$ as n gets very large?
3. What problems arise when you try to determine the 200th term?
4. Use the results of step 1 and step 2 to establish an **exact** value for the continued fraction.

5. Now consider another continued fraction.

$$\begin{array}{c} 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\dots}}}}}}} \end{array}$$

Repeat steps 1 to 4 using this continued fraction.

6. Now consider the general continued fraction.

$$\begin{array}{c} k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{\dots}}}}}} \end{array}$$

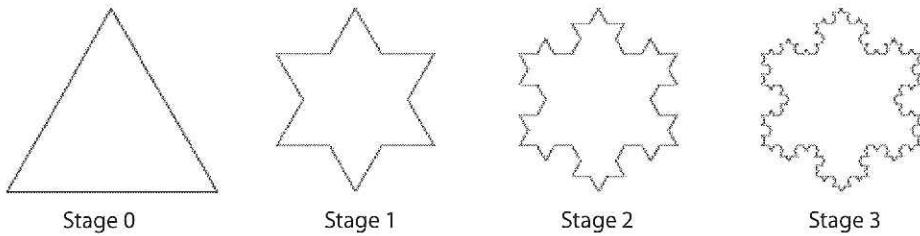
By considering other values of k , determine a generalized statement for the exact value of any such continued fraction. For which values of k does your generalized statement hold true? How do you know? Provide evidence to support your answer.

2 The Koch snowflake

SL Type I

Description

In 1904 Helge von Koch identified a fractal that appeared to model the snowflake. The fractal is built by starting with an equilateral triangle and removing the inner third of each side, building another equilateral triangle at the location where the side was removed, and then repeating the process indefinitely. The process is illustrated clearly below, showing the original triangle at stage 0 and the resulting figures after one, two and three iterations.



Method

Let N_n = the number of sides, I_n = the length of a single side, P_n = the length of the perimeter and A_n = the area of the snowflake, at the n^{th} stage.

1. Using an initial side length of 1, create a table that shows the values of N_n , I_n , P_n and A_n for $n = 0$, 1, 2 and 3. Use exact values in your results. Explain the relationship between successive terms in the table for each quantity N_n , I_n , P_n and A_n .
2. Using a GDC or a suitable graphing software package, create graphs of the four sets of values plotted against the value of n . Provide separate printed output for each graph.
3. For each of the graphs above, develop a statement in terms of n that generalizes the behaviour shown in its graph. Explain how you arrived at your generalizations. Verify that your generalizations apply consistently to the sets of values produced in the table.
4. Investigate what happens at $n = 4$. Use your conjectures from step 3 to obtain values for N_4 , I_4 , P_4 and A_4 . Now draw a large diagram of one “side” (that is, one side of the original triangle that has been transformed) of the fractal at stage 4 and clearly verify your predictions.
5. Calculate values for N_6 , I_6 , P_6 and A_6 . You need not verify these answers.
6. Write down successive values of A_n in terms of A_0 . What pattern emerges?
7. Explain what happens to the perimeter and area as n gets very large. What conclusion can you make about the area as $n \rightarrow \infty$? Comment on your results.

3 Matrix powers

SL Type I

Method

- Consider the matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Calculate M^n for $n = 2, 3, 4, 5, 10, 20, 50$.

Describe in words any pattern you observe.

Use this pattern to find a general expression for the matrix M^n in terms of n .

- Consider the matrices $P = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $S = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$.

$$P^2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2 \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}; S^2 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^2 = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2 \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

Calculate P^n and S^n for other values of n and describe any pattern(s) you observe.

- Now consider matrices of the form $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$.

Steps 1 and 2 contain examples of these matrices for $k = 1, 2$ and 3 .

Consider other values of k , and describe any pattern(s) you observe.

Generalize these results in terms of k and n .

- Use technology to investigate what happens with further values of k and n . State the scope or limitations of k and n .
- Explain why your results hold true in general.

Type I tasks

Matrix powers—student A

SL Type I

Matrix Powers

(Lacks an introduction)

$$1. \quad M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{aligned} M^2 &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4+0 & 0+0 \\ 0+0 & 0+4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \end{aligned}$$

Using GDC;

$$M^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}, \quad M^4 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$$

$$M^5 = \begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}, \quad M^{10} = \begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$$

$$M^{20} = \begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$$

$$M^{50} = \begin{pmatrix} 1.13 \times 10^{15} & 0 \\ 0 & 1.13 \times 10^{15} \end{pmatrix}$$

For each higher power of M the entries double.

$$\text{So } M^2 = 2 \cdot M$$

$$M^3 = 2 \cdot M^2 = 2^2 \cdot M$$

$$M^4 = 2 \cdot M^3 = 2^3 \cdot M$$

⋮

$$M^n = 2^{n-1} \cdot M$$

- 1 -

$$2. \quad P = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2 \cdot \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

just enough
data

$$\text{by GDC } P^3 = \begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix} = 2 \begin{pmatrix} 18 & 14 \\ 14 & 18 \end{pmatrix} = 2^2 \begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix} = 2 \begin{pmatrix} 68 & 60 \\ 60 & 68 \end{pmatrix}$$

$$= 2^3 \begin{pmatrix} 34 & 30 \\ 30 & 34 \end{pmatrix} = 2^3 \begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix}$$

I will continue taking powers of 2 out.

$$P^5 = \begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix} = 2^4 \begin{pmatrix} 33 & 31 \\ 31 & 33 \end{pmatrix}$$



It seems that P^n will have a factor 2^{n-1} . If we look at Part 4, this gives a hint for the structure of the remaining matrices.

$$\text{i.e. } \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2+1 & 2-1 \\ 2-1 & 2+1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2^2+1 & 2^2-1 \\ 2^2-1 & 2^2+1 \end{pmatrix}$$

just
enough
data

$$\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 2^3+1 & 2^3-1 \\ 2^3-1 & 2^3+1 \end{pmatrix}$$

- 2 -

2. cont'd

$$\begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix} = \begin{pmatrix} 2^4+1 & 2^{4-1} \\ 2^{4-1} & 2^4+1 \end{pmatrix}$$

Therefore it appears that the pattern is;

$$P^n = 2^{n-1} \begin{pmatrix} 2^n+1 & 2^{n-1} \\ 2^{n-1} & 2^n+1 \end{pmatrix}$$

Check!

$$\begin{aligned} P^{10} &= \begin{pmatrix} 524800 & 523776 \\ 523776 & 524800 \end{pmatrix} \\ (\text{by GDC}) &= 512 \begin{pmatrix} 1025 & 1023 \\ 1023 & 1025 \end{pmatrix} \\ &= 2^9 \begin{pmatrix} 2^{10}+1 & 2^{10}-1 \\ 2^{10}-1 & 2^{10}+1 \end{pmatrix} \end{aligned}$$

$$\text{For } S = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\begin{aligned} (\text{by GDC}) \quad S^2 &= \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2 \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} \end{aligned}$$

not
enough
data
generated

$$S^3 = \begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix} = 4 \begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix}$$

This looks different, as we could take out another factor of 2.

- 3 -

2. Cont'd

$$\text{However, } \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3+1 & 3-1 \\ 3-1 & 3+1 \end{pmatrix}$$

$$\text{so in } \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} \quad k=3$$

$$\text{Now } S^2 = 2 \cdot \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3^2+1 & 3^2-1 \\ 3^2-1 & 3^2+1 \end{pmatrix}$$

$$S^3 = 4 \begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix} = 2^2 \begin{pmatrix} 3^3+1 & 3^3-1 \\ 3^3-1 & 3^3+1 \end{pmatrix}$$

So it appears that the pattern might be;

$$S^n = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$$

Check!

$$\begin{aligned} (by GDC) \quad S^5 &= \begin{pmatrix} 3964 & 3872 \\ 3872 & 3904 \end{pmatrix} \\ &= 16 \begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix} \\ &= 2^4 \begin{pmatrix} 3^5+1 & 3^5-1 \\ 3^5-1 & 3^5+1 \end{pmatrix} \end{aligned}$$

So, in general

$$S^n = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$$
Correct

- 4 -

3. For $M = \begin{pmatrix} K+1 & K-1 \\ K-1 & K+1 \end{pmatrix}$

we expect

$$M^n = 2^{n-1} \begin{pmatrix} K^n+1 & K^n-1 \\ K^n-1 & K^n+1 \end{pmatrix}$$

4. Using the GDC I will try some other values of K and n .

$K=5, n=7$

$$M^7 = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}^7$$

Only one further example checked
No discussion of scope or limitations

by the pattern, this should be;

$$M^7 = 2^6 \begin{pmatrix} 5^7+1 & 5^7-1 \\ 5^7-1 & 5^7+1 \end{pmatrix}$$

$$2^6 = 64 \quad 5^7+1 = 78125+1 = 78126 \\ 5^7-1 = 78125-1 = 78124$$

by GDC;

$$\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}^7 = \begin{pmatrix} 5000064 & 4999936 \\ 4999936 & 5000064 \end{pmatrix}$$

$$= 64 \begin{pmatrix} 78126 & 78124 \\ 78124 & 78126 \end{pmatrix}$$

$$= 2^6 \begin{pmatrix} 5^7+1 & 5^7-1 \\ 5^7-1 & 5^7+1 \end{pmatrix}$$

- 5 -

5. Consider $M = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$

$$\begin{aligned}
 M^2 &= \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} \\
 &= \begin{pmatrix} (k+1)^2 + (k-1)^2 & (k+1)(k-1) + (k-1)(k+1) \\ (k-1)(k+1) + (k+1)(k-1) & (k-1)^2 + (k+1)^2 \end{pmatrix} \\
 &= \begin{pmatrix} k^2 + 2k + 1 + k^2 - 2k + 1 & k^2 - 1 + k^2 - 1 \\ k^2 - 1 + k^2 - 1 & k^2 + 2k + 1 + k^2 - 2k + 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2k^2 + 2 & 2k^2 - 2 \\ 2k^2 - 2 & 2k^2 + 2 \end{pmatrix} \\
 &= 2 \begin{pmatrix} k^2 + 1 & k^2 - 1 \\ k^2 - 1 & k^2 + 1 \end{pmatrix}, \text{ which fits the pattern.}
 \end{aligned}$$

Now

$$\begin{aligned}
 M^3 &= M^2 \cdot M = 2 \begin{pmatrix} k^2 + 1 & k^2 - 1 \\ k^2 - 1 & k^2 + 1 \end{pmatrix} \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} \\
 &= 2 \begin{pmatrix} 2k^3 + 2 & 2k^3 - 2 \\ 2k^3 - 2 & 2k^3 + 2 \end{pmatrix} \\
 &= 2^2 \begin{pmatrix} k^3 + 1 & k^3 - 1 \\ k^3 - 1 & k^3 + 1 \end{pmatrix}
 \end{aligned}$$

So, the pattern works !!

- 6 -

This page provides a suitable informal justification.

Mathematics SL: The portfolio		Form B
Feedback to student		
Name: Student A		
Title of task: Matrix powers		Type: <input checked="" type="radio"/> I <input type="radio"/> II
Date set: 12/03/05	Date submitted: 20/03/05	
A Use of notation and terminology		2 / 2
<p>Successful analysis. Only just enough data to warrant level above 2. Only one further example looked at, not enough for 5.</p>		
B Communication		2 / 3
No introduction, more explanation needed. Hard to follow in places.		
C Mathematical process		4 / 5
<p>A lack of any discussion of scope or limitations prevents the award of a higher level, even though an informal justification is given.</p>		
D Results		3 / 5
<p>Not fully exploited. The GDC allows for fuller explanation of other cases.</p>		
E Use of technology		2 / 3
<p>Satisfactory</p>		
F Quality of work		1 / 2

Matrix powers—student B

SL Type I

1. This assignment is looking for a general formula that will give the n th power of the matrix $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$.

The simplest matrix of this form is

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This table gives results for various values of n .

n	M^n
1	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 2^2 \end{pmatrix}$
3	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2^3 & 0 \\ 0 & 2^3 \end{pmatrix}$
4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 2^4 & 0 \\ 0 & 2^4 \end{pmatrix}$
5	$\begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix} = \begin{pmatrix} 2^5 & 0 \\ 0 & 2^5 \end{pmatrix}$
10	$\begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}$
20	$\begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix} = \begin{pmatrix} 2^{20} & 0 \\ 0 & 2^{20} \end{pmatrix}$

How? GDC? \boxed{E}
Solve so?

The elements of the leading diagonal are powers of 2. The other two elements are zero in every case.

$$M^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix} \quad \checkmark$$

2. The table gives result for $P^n = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^n$ and

$$S^n = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^n$$

n	P^n	S^n
1	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
2	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$	$\begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$
3	$\begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$	$\begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix}$
4	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$	$\begin{pmatrix} 656 & 640 \\ 640 & 656 \end{pmatrix}$
5	$\begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix}$	$\begin{pmatrix} 3904 & 3872 \\ 3872 & 3904 \end{pmatrix}$

how? (1)
GCD? again
again, so?
so?

It is more difficult to spot any patterns in these results. To help, the numbers obtained are broken down in the tables below

For P

n	1st element	2nd element
1	3	1
2	$10 = 3 \times 3 + 1$	$6 = 3 + 1 \times 3$
3	$36 = 10 \times 3 + 6$	$28 = 10 + 6 \times 3$
4	$136 = 36 \times 3 + 28$	$120 = 36 + 28 \times 3$
5	$528 = 136 \times 3 + 120$	$496 = 136 + 120 \times 3$

$$\text{Let } P^n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}$$

then these patterns can be written as

$$\begin{aligned}
 a_1 &= a_1 & b_1 &= b_1 \\
 a_2 &= 3a_1 + b_1 & b_2 &= a_1 + 3b_1 \\
 a_3 &= 3a_2 + b_2 & b_3 &= a_2 + 3b_2 \\
 a_4 &= 3a_3 + b_3 & b_4 &= a_3 + 3b_3
 \end{aligned}$$

which leads to the general formula

$$P^n = \begin{pmatrix} 3a_{n-1} + b_{n-1} & a_{n-1} + 3b_{n-1} \\ a_{n-1} + 3b_{n-1} & 3a_{n-1} + b_{n-1} \end{pmatrix}$$

For S

n	1st element	2nd element
1	4	2
2	$20 = 4 \times 4 + 2 \times 2$	$16 = 4 \times 2 + 2 \times 4$
3	$112 = 20 \times 4 + 16 \times 2$	$104 = 20 \times 2 + 16 \times 4$
4	$656 = 112 \times 4 + 104 \times 2$	$640 = 112 \times 2 + 104 \times 4$
5	$3904 = 656 \times 4 + 640 \times 2$	$3872 = 656 \times 2 + 640 \times 4$

This generalises to

$$S^n = \begin{pmatrix} 4a_{n-1} + 2b_{n-1} & 2a_{n-1} + 4b_{n-1} \\ 2a_{n-1} + 4b_{n-1} & 4a_{n-1} + 2b_{n-1} \end{pmatrix}$$

$$3. \text{ If } k=4, R = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = R$$

$$\begin{aligned}
 R^n &= R^{n-1}R \\
 &= \begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \quad \checkmark \text{ good analysis!} \\
 &= \begin{pmatrix} 5a_{n-1} + 3b_{n-1} & 3a_{n-1} + 5b_{n-1} \\ 3a_{n-1} + 5b_{n-1} & 5a_{n-1} + 3b_{n-1} \end{pmatrix} \quad \textcircled{C}
 \end{aligned}$$

If $k = 5$ the matrix is $\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix} = Q$

$$\begin{aligned} Q^n &= Q^{n-1} Q \\ &= \begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 6a_{n-1} + 4b_{n-1} & 4a_{n-1} + 6b_{n-1} \\ 4a_{n-1} + 6b_{n-1} & 6a_{n-1} + 4b_{n-1} \end{pmatrix} \end{aligned}$$

This suggests the general result

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = \begin{pmatrix} (k+1)a_{n-1} + (k-1)b_{n-1} & (k-1)a_{n-1} + (k+1)b_{n-1} \\ (k-1)a_{n-1} + (k+1)b_{n-1} & (k+1)a_{n-1} + (k-1)b_{n-1} \end{pmatrix}$$

4. Let $k = 2, n = -1$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \quad \text{how did you obtain } \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1} ? \text{ (B)}$$

Let $k = 2, n = -2$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-2} = \begin{pmatrix} \frac{5}{32} & -\frac{3}{32} \\ -\frac{3}{32} & \frac{5}{32} \end{pmatrix}$$

$$3 \cdot \frac{5}{32} + -\frac{3}{32} = \frac{12}{32} = \frac{3}{8}$$

$$\frac{5}{32} + 3 \left(-\frac{3}{32} \right) = -\frac{4}{32} = -\frac{1}{8}$$

So the result also holds for negative values of n .

$$n = 0, \quad \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$3 \times 1 + 0 = 3 \quad 1 + 3 \times 0 = 1$$

So the result holds for $n=1$

$$\frac{3 \times 3}{8} + -\frac{1}{8} = \frac{8}{8} = 1$$

$$\frac{3}{8} + 3\left(-\frac{1}{8}\right) = 0 = 0$$

So result also holds for $n=0$.

$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n$ is undefined if n is not an integer.

The result holds for $n \in \mathbb{Z}$

Suppose k is negative

let $k=-1$ giving the matrix $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

and let $k=-2$ giving the matrix $\begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}$

The table shows the first few results

n	$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}^n$	$\begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}^n$
1	$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 10 & 6 \\ 6 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$
3	$\begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$	$\begin{pmatrix} -28 & -36 \\ -36 & -28 \end{pmatrix}$
4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$
5	$\begin{pmatrix} -32 & -32 \\ -32 & 0 \end{pmatrix}$	$\begin{pmatrix} -496 & -528 \\ -528 & -496 \end{pmatrix}$

These results fit the general formula so it seems that the result holds for negative values of k .

Let $k = \frac{1}{2}$

$$\begin{array}{c} n \\ \left(\begin{array}{cc} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{array} \right)^n \\ 1 \\ \left(\begin{array}{cc} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{array} \right) \\ 2 \\ \left(\begin{array}{cc} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{array} \right) \\ 3 \\ \left(\begin{array}{cc} \frac{9}{2} & -\frac{7}{2} \\ -\frac{7}{2} & \frac{9}{2} \end{array} \right) \\ 4 \\ \left(\begin{array}{cc} \frac{17}{2} & -\frac{15}{2} \\ -\frac{15}{2} & \frac{17}{2} \end{array} \right) \end{array}$$

good discussion
of scope and
limitations! \heartsuit

Substituting into the general formula

$$\frac{3}{2} \cdot \frac{3}{2} + -\frac{1}{2} \cdot -\frac{1}{2} = \frac{9}{4} + \frac{1}{4} = \frac{5}{2}$$

$$-\frac{1}{2} \cdot \frac{3}{2} + \frac{3}{2} \cdot -\frac{1}{2} = -\frac{6}{4} = \frac{3}{2}$$

suggests that the result will also hold for rational values of k .

Let $k = \sqrt{2}$

$$\begin{array}{c} n \\ \left(\begin{array}{cc} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{array} \right)^n \\ 1 \\ \left(\begin{array}{cc} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{array} \right) \\ 2 \\ \left(\begin{array}{cc} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{array} \right) \left(\begin{array}{cc} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{array} \right) = \boxed{2\sqrt{3}+3} \\ 3 \end{array}$$

pursue this:
KFR
works! \heartsuit

The result holds for $n \in \mathbb{Z}$ and k rational.

5. This is easily justified by matrix multiplication.

$$\begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} (k+1) & (k-1) \\ (k-1) & (k+1) \end{pmatrix}$$

$$= \begin{pmatrix} (k+1)a_{n-1} + (k-1)b_{n-1} & (k-1)a_{n-1} + (k+1)b_{n-1} \\ (k-1)a_{n-1} + (k+1)b_{n-1} & (k+1)a_{n-1} + (k-1)b_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} (k+1) & (k-1) \\ (k-1) & (k+1) \end{pmatrix}^n.$$

satisfactory
informal
justification \textcircled{D}

Mathematics SL: The portfolio		Form B
Feedback to student		
Name: Student B		
Title of task: Matrix powers		Type: <input checked="" type="radio"/> I <input type="radio"/> II
Date set:	Date submitted:	
A Use of notation and terminology		2 / 2
Good use of appropriate notation throughout.		
B Communication		3 / 3
Well presented. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$ result appears without working or explanation. Overall communication, however, is clear and coherent.		
C Mathematical process		5 / 5
Good analysis. Validity tested.		
D Results		5 / 5
While the recursive statement does not give an explicit result, it is an acceptable general statement. Discussion of scope and limitations is incomplete, but correct, satisfactory informal justification.		
E Use of technology		2 / 3
No evidence of use is provided. Limited use only. You could have explored irrational values of k with decimal approximations.		
F Quality of work		1 / 2
Good work. An explicit generalized statement and consideration of irrational values of k would have demonstrated greater insight.		