

When $x = 6$, $y = -6^2 + 12(6) - 20 = -36 + 72 - 20 = 16$

∴ the maximum value of $y = -x^2 + 12x - 20$ is 16, which occurs when $x = 6$.

- b i The perimeter is 20

$$\therefore x + y + 8 = 20$$

$$\therefore y = 12 - x$$

- iii Since $y = 12 - x$, $(12 - x)^2 = x^2 + 64 - 16x \cos \theta$

$$\therefore 144 - 24x + x^2 = x^2 + 64 - 16x \cos \theta$$

$$\therefore 16x \cos \theta = 24x - 80$$

$$\therefore \cos \theta = \frac{24x - 80}{16x}$$

$$= \frac{3x - 10}{2x}$$

- c Area $A = \frac{1}{2} \times x \times 8 \times \sin \theta$

$$= 4x \sin \theta$$

$$\therefore A^2 = 16x^2 \sin^2 \theta$$

$$= 16x^2(1 - \cos^2 \theta)$$

$$= 16x^2 \left[1 - \left(\frac{3x - 10}{2x} \right)^2 \right]$$

$$= 16x^2 \left[1 - \frac{9x^2 - 60x + 100}{4x^2} \right]$$

$$= 16x^2 - 4(9x^2 - 60x + 100)$$

$$= 16x^2 - 36x^2 + 240x - 400$$

$$= -20x^2 + 240x - 400$$

$$= 20(-x^2 + 12x - 20)$$

- 9 a $QS^2 = 6^2 + 3^2 - 2 \times 6 \times 3 \times \cos \phi$

$$\therefore QS = \sqrt{45 - 36 \cos \phi}$$

- b i If $\phi = 50^\circ$, $QS = \sqrt{45 - 36 \cos 50^\circ} \approx 4.675$

$$\therefore \frac{\sin \theta}{7} \approx \frac{\sin 32^\circ}{4.675}$$

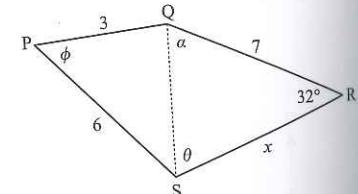
$$\therefore \sin \theta \approx \frac{7 \times \sin 32^\circ}{4.675}$$

$$\therefore \theta \approx \sin^{-1} \left(\frac{7 \times \sin 32^\circ}{4.675} \right) \text{ or its supplement}$$

$$\therefore \theta \approx 52.5^\circ \text{ or } (180 - 52.5)^\circ$$

$$\therefore \widehat{RSQ} \approx 52.5^\circ \text{ or } 127.5^\circ$$

- ii If θ is acute, then $\theta \approx 52.5^\circ$ and $\alpha = 180^\circ - 32^\circ - \theta \approx 95.5^\circ$



$$\therefore \frac{x}{\sin 95.5^\circ} \approx \frac{7}{\sin 52.5^\circ}$$

$$\therefore x \approx \frac{7 \times \sin 95.5^\circ}{\sin 52.5^\circ}$$

$$\therefore x \approx 8.78$$

$$\therefore \text{perimeter} \approx 6 + 3 + 7 + 8.78 \approx 24.8 \text{ units}$$

- c area of PQRS = $\frac{1}{2} \times 3 \times 6 \times \sin 50^\circ + \frac{1}{2} \times 7 \times 8.78 \times \sin 32^\circ \approx 23.2 \text{ units}^2$

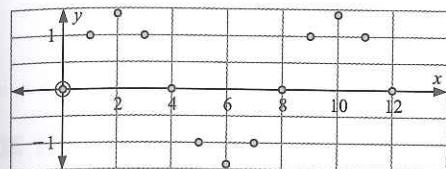
or if θ is obtuse we can similarly calculate that the area of PQRS $\approx 12.6 \text{ units}^2$.

Chapter 12

ADVANCED TRIGONOMETRY

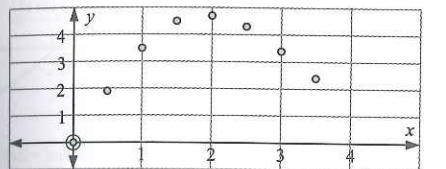
EXERCISE 12A

1 a



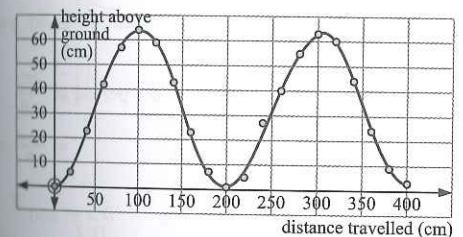
Data exhibits periodic behaviour.

c



Not enough information to say data is periodic. It may in fact be quadratic.

2 a



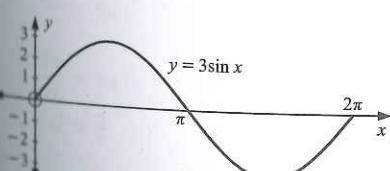
c A curve can be fitted to the data as the distance travelled is continuous.

- 1 a periodic b periodic c periodic d not periodic e periodic f periodic

EXERCISE 12B.1

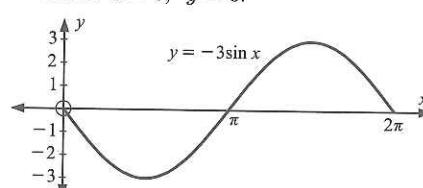
1 a $y = 3 \sin x$

has amplitude 3 and period $\frac{2\pi}{1} = 2\pi$
When $x = 0$, $y = 0$.



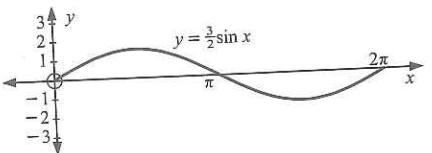
b $y = -3 \sin x$

has amplitude |-3| = 3
and period $\frac{2\pi}{1} = 2\pi$.
When $x = 0$, $y = 0$.

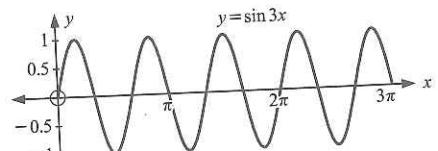


It is the reflection of $y = 3 \sin x$ in the x-axis.

- c $y = \frac{3}{2} \sin x$
has amplitude $\frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.
When $x = 0$, $y = 0$.

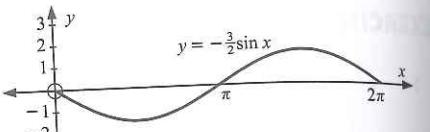


- 2 a $y = \sin 3x$
has amplitude 1 and period $\frac{2\pi}{3}$.
When $x = 0$, $y = 0$.



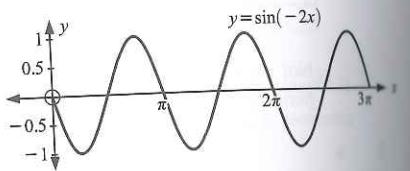
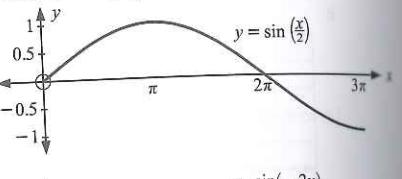
- c $y = \sin(-2x)$
has amplitude 1 and period $\frac{2\pi}{|-2|} = \pi$.
When $x = 0$, $y = 0$.

- d $y = -\frac{3}{2} \sin x$
has amplitude $|\frac{-3}{2}| = \frac{3}{2}$
and period $\frac{2\pi}{1} = 2\pi$.



It is the reflection of $y = \frac{3}{2} \sin x$ in the x -axis.

- b $y = \sin(\frac{x}{2})$
has amplitude 1 and period $\frac{2\pi}{\frac{1}{2}} = 4\pi$.
When $x = 0$, $y = 0$.

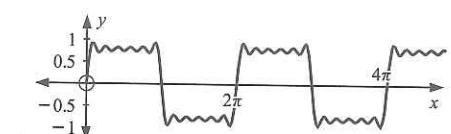
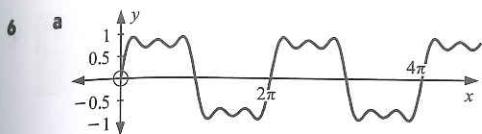
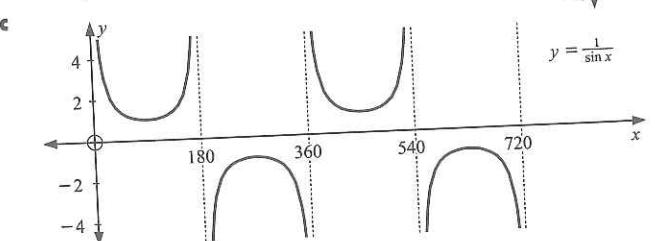
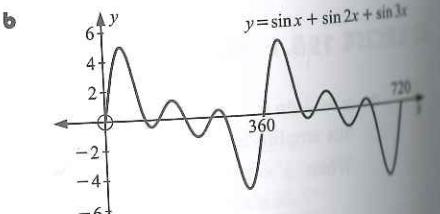
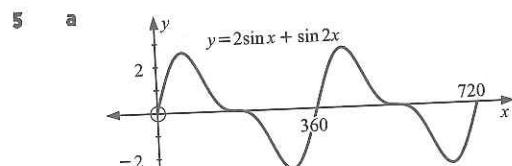


It is the reflection of $y = \sin 2x$ in the y -axis.

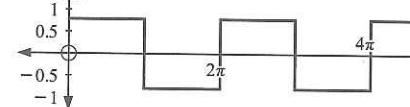
3 a period = $\frac{2\pi}{|4|}$ b period = $\frac{2\pi}{|-4|}$
 $= \frac{\pi}{2}$ $= \frac{\pi}{2}$

c period = $\frac{2\pi}{|\frac{1}{3}|}$ d period = $\frac{2\pi}{0.6}$
 $= 6\pi$ $= \frac{20\pi}{6} = \frac{10\pi}{3}$

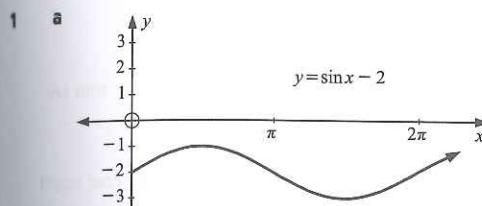
4 a $\frac{2\pi}{B} = 5\pi$ b $\frac{2\pi}{B} = \frac{2\pi}{3}$ c $\frac{2\pi}{B} = 12\pi$ d $\frac{2\pi}{B} = 4$ e $\frac{2\pi}{B} = 100$
 $\therefore B = \frac{2}{5}$ $\therefore B = 3$ $\therefore B = \frac{1}{6}$ $\therefore B = \frac{\pi}{2}$ $\therefore B = \frac{2\pi}{100} = \frac{\pi}{50}$



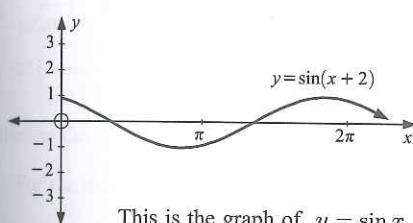
Prediction:



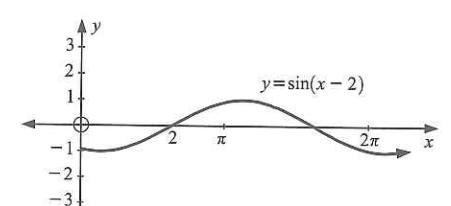
EXERCISE 12B.2



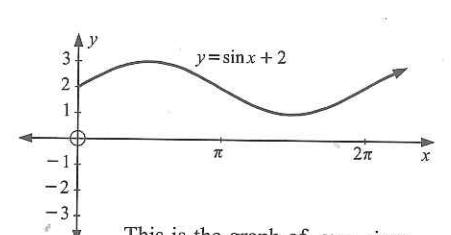
This is the graph of $y = \sin x$ translated by $(0, -2)$.



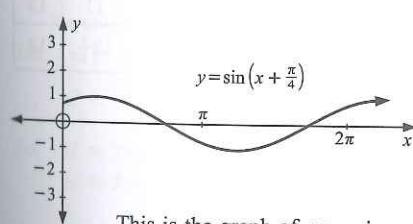
This is the graph of $y = \sin x$ translated by $(-2, 0)$.



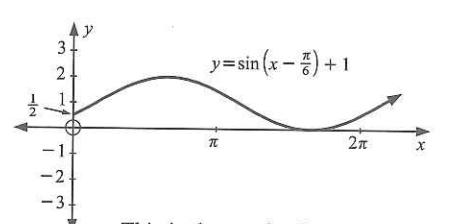
This is the graph of $y = \sin x$ translated by $(2, 0)$.



This is the graph of $y = \sin x$ translated by $(0, 2)$.



This is the graph of $y = \sin x$ translated by $(-\frac{\pi}{4}, 0)$.



This is the graph of $y = \sin x$ translated by $(\frac{\pi}{6}, 1)$.

a period = $\frac{2\pi}{|5|} = \frac{2\pi}{5}$ b period = $\frac{2\pi}{|\frac{1}{4}|} = 8\pi$

c period = $\frac{2\pi}{|-2|} = \pi$
 $\frac{2\pi}{B} = 3\pi$ b $\frac{2\pi}{B} = \frac{\pi}{10}$ c $\frac{2\pi}{B} = 100\pi$ d $\frac{2\pi}{B} = 50$
 $\therefore B = \frac{2}{3}$ $\therefore B = 20$ $\therefore B = \frac{2}{100} = \frac{1}{50}$ $\therefore B = \frac{2\pi}{50} = \frac{\pi}{25}$

A translation of $(0, -1)$, or vertically down 1 unit.

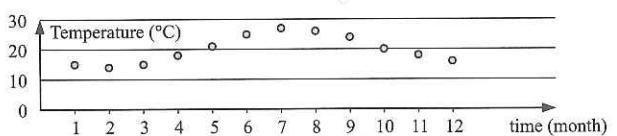
b A translation of $(\frac{\pi}{4}, 0)$, or horizontally $\frac{\pi}{4}$ units right.

- c A vertical stretch of factor 2.
 e A vertical stretch of factor $\frac{1}{2}$.
 g A reflection in the x -axis.
 i A vertical stretch of factor 2 followed by a horizontal compression of factor 3.
 j A translation of $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.
- d A horizontal compression of factor 4.
 f A horizontal stretch of factor 4.
 h A translation of $(-2, -3)$.

EXERCISE 12C

1

	Month, t	1	2	3	4	5	6	7	8	9	10	11	12
	Temp, T	15	14	15	18	21	25	27	26	24	20	18	16



The period is 12 months so $\frac{2\pi}{B} = 12 \therefore B = \frac{\pi}{6}$ {assuming $B > 0$ }.

Amplitude, $A \approx \frac{\text{max.} - \text{min.}}{2} \approx \frac{27 - 14}{2} \approx 6.5$

As the principal axis is midway between min. and max., then $D \approx \frac{27 + 14}{2} \approx 20.5$

When T is 20.5 (midway between min. and max.)

$$C \approx \frac{2 + 7}{2} \approx 4.5 \quad \text{average of } t \text{ values}$$

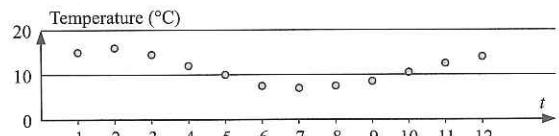
$$\therefore T \approx 6.5 \sin \frac{\pi}{6}(t - 4.5) + 20.5 \quad \text{where } \frac{\pi}{6} \approx 0.524.$$

b Using technology, $T \approx 6.14 \sin(0.575t - 2.70) + 20.4$

$$\therefore T \approx 6.14 \sin 0.575(t - 4.70) + 20.4$$

2

	Month, t	1	2	3	4	5	6	7	8	9	10	11	12
	Temp, T	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14



The period is $\frac{2\pi}{B} = 12 \therefore B = \frac{\pi}{6}$ {assuming $B > 0$ }

Amplitude, $A \approx \frac{\text{max.} - \text{min.}}{2} \approx \frac{16 - 7}{2} \approx 4.5$

As the principal axis is midway between min. and max., then $D \approx \frac{16 + 7}{2} \approx 11.5$

At min., $t = 7$ and at max., $t = 2 + 12 = 14 \therefore C = \frac{7 + 14}{2} = 10.5$

$$\text{So, } T \approx 4.5 \sin \frac{\pi}{6}(t - 10.5) + 11.5$$

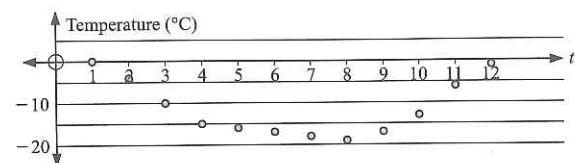
b Using tech., $T \approx 4.29 \sin(0.533t + 0.769) + 11.2$

$$\therefore T \approx 4.29 \sin 0.533(t + 1.44) + 11.2$$

Note: (1) $\frac{\pi}{6} \approx 0.524 \checkmark$

$$(2) 1.44 - (-10.5) = 11.94 \approx 12$$

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	0	-4	-10	-15	-16	-17	-18	-19	-17	-13	-6	-1



The period is $\frac{2\pi}{B} = 12 \therefore B = \frac{\pi}{6}$ {assuming $B > 0$ }

Amplitude, $A \approx \frac{\text{max.} - \text{min.}}{2} \approx \frac{0 - (-19)}{2} \approx 9.5$

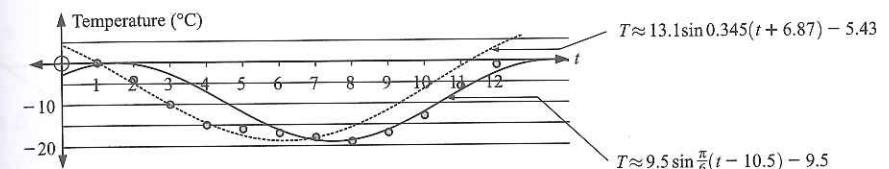
$D \approx \frac{\text{max.} + \text{min.}}{2} \approx \frac{0 + (-19)}{2} \approx -9.5$

At min., $t = 8$ and at max., $t = 1 + 12 = 13 \therefore C \approx \frac{8 + 13}{2} \approx 10.5$

$$\text{So, } T \approx 9.5 \sin \frac{\pi}{6}(t - 10.5) - 9.5 \quad \dots \quad (1)$$

From technology, $T \approx 13.1 \sin(0.345t + 2.37) - 5.43$

$$\therefore T \approx 13.1 \sin 0.345(t + 6.87) - 5.43 \quad \dots \quad (2)$$



Neither model seems appropriate.

4 a For the model $H = A \sin B(t - C) + D$

$$\text{period} = \frac{2\pi}{B} = 12.4 \text{ hours} \therefore B = \frac{2\pi}{12.4} \approx 0.507$$

We let the principal axis be 0, so $D = 0$

\therefore the amplitude $A = 7$, so the min. is -7, and the max. is +7

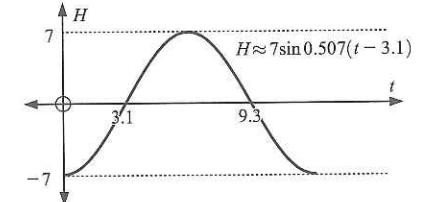
Let $t = 0$ correspond to 'low tide' $\therefore t = 6.2$ corresponds to 'high tide'

$$\therefore C = \frac{0 + 6.2}{2} = 3.1$$

$$\text{So, } H \approx 7 \sin 0.507(t - 3.1) + 0$$

$$\therefore H \approx 7 \sin 0.507(t - 3.1)$$

b



5 Let the model be $H = A \sin B(t - C) + D$ metres

When $t = 0$, $H = 2$ and when $t = 50$, $H = 22$

$$\begin{matrix} \uparrow \\ \text{min.} \end{matrix} \qquad \qquad \begin{matrix} \uparrow \\ \text{max.} \end{matrix}$$

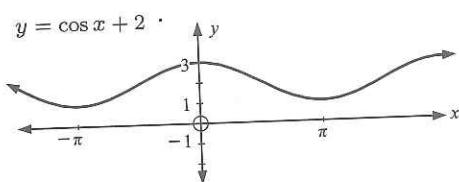
$$\text{period} = \frac{2\pi}{B} = 100 \therefore B = \frac{2\pi}{100} = \frac{\pi}{50}$$

$$A = 10 \quad \{\text{from the diagram}\} \quad D = \frac{\text{max.} + \text{min.}}{2} = \frac{22 + 2}{2} = 12$$

$$C = \frac{0 + 50}{2} = 25 \quad \{\text{values of } t \text{ at min. and max.}\} \quad \therefore H = 10 \sin \frac{\pi}{50}(t - 25) + 12$$

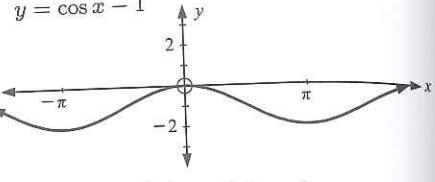
EXERCISE 12D

1 a $y = \cos x + 2$



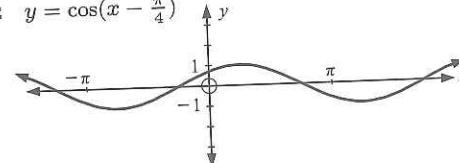
This is a vertical translation of $y = \cos x$ through $(0, 2)$.

b $y = \cos x - 1$



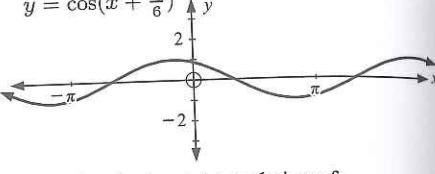
This is a vertical translation of $y = \cos x$ through $(0, -1)$.

c $y = \cos(x - \frac{\pi}{4})$



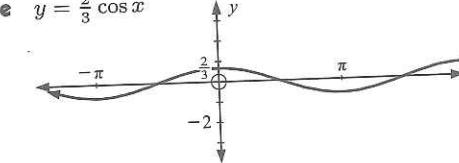
This is a horizontal translation of $y = \cos x$ through $(\frac{\pi}{4}, 0)$.

d $y = \cos(x + \frac{\pi}{6})$



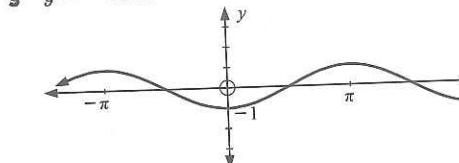
This is a horizontal translation of $y = \cos x$ through $(-\frac{\pi}{6}, 0)$.

e $y = \frac{2}{3} \cos x$



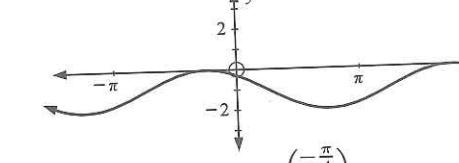
This is a vertical stretch of $y = \cos x$ with factor $\frac{2}{3}$.

g $y = -\cos x$



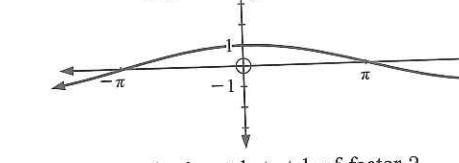
This is a reflection of $y = \cos x$ in the x-axis.

i $y = \cos(x + \frac{\pi}{4}) - 1$



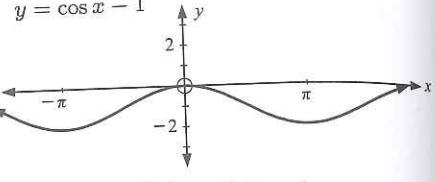
This is a translation of $(-\frac{\pi}{4}, -1)$.

k $y = \cos(\frac{x}{2})$



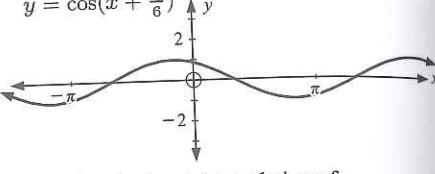
This is a horizontal stretch of factor 2.

b $y = \cos x - 1$



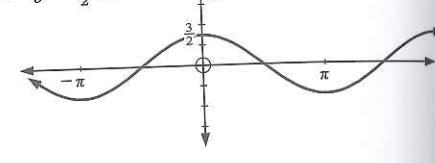
This is a vertical translation of $y = \cos x$ through $(0, -1)$.

d $y = \cos(x + \frac{\pi}{6})$



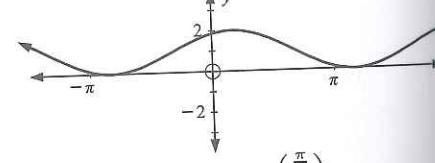
This is a horizontal translation of $y = \cos x$ through $(-\frac{\pi}{6}, 0)$.

f $y = \frac{3}{2} \cos x$



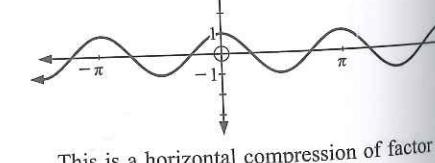
This is a vertical stretch of $y = \cos x$ with factor $\frac{3}{2}$.

h $y = \cos(x - \frac{\pi}{6}) + 1$



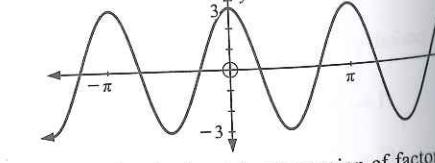
This is a translation of $(\frac{\pi}{6}, 1)$.

j $y = \cos(2x)$



This is a horizontal compression of factor 2.

l $y = 3 \cos(2x)$



This is a horizontal compression of factor 2 followed by a vertical stretch of factor 3.

2 a period = $\frac{2\pi}{3}$

b period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$

c period = $\frac{2\pi}{\frac{\pi}{50}} = 100$

3 A controls the amplitude. B controls the period {period = $\frac{2\pi}{|B|}$ }. C controls the horizontal translation. D controls the vertical translation.

4 a If $y = A \cos B(x - C) + D$, then $A = 2$, $\pi = \frac{2\pi}{B}$ ∴ $B = 2$
 C and D are 0 as there is no horizontal or vertical shift. ∴ $y = 2 \cos(2x)$

b If $y = A \cos B(x - C) + D$, then $A = 1$, $4\pi = \frac{2\pi}{B}$ ∴ $B = \frac{1}{2}$
A vertical shift of 2 units, no horizontal shift ∴ $D = 2$, $C = 0$.
So, $y = \cos(\frac{1}{2}x) + 2$ or $y = \cos(\frac{\pi}{2}x) + 2$.

c If $y = A \cos B(x - C) + D$, then $A = -5$, $6 = \frac{2\pi}{B}$ ∴ $B = \frac{\pi}{3}$
 $C = D = 0$ {as there is no translation} ∴ $y = -5 \cos(\frac{\pi}{3}x)$

EXERCISE 12E.1

1 a $\tan 0^\circ$

b $\tan 15^\circ \approx 0.268$

c $\tan 20^\circ \approx 0.364$

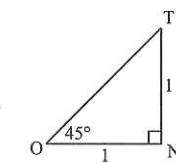
d $\tan 25^\circ \approx 0.466$

e $\tan 35^\circ \approx 0.700$

f $\tan 45^\circ = 1$

g $\tan 50^\circ \approx 1.19$

h $\tan 55^\circ \approx 1.43$

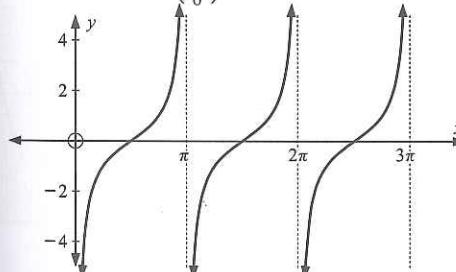


2 In ΔTON, ON = NT = 1 {Δ is isosceles} $\tan 45^\circ = \frac{NT}{ON} = \frac{1}{1} = 1$

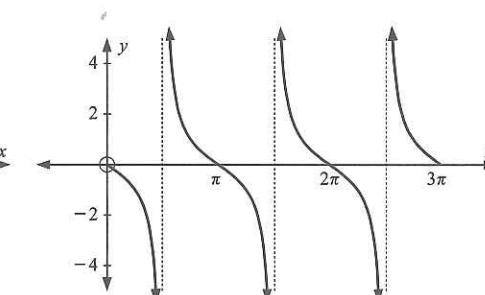
EXERCISE 12E.2

1 a i $y = \tan(x - \frac{\pi}{2})$ is $y = \tan x$

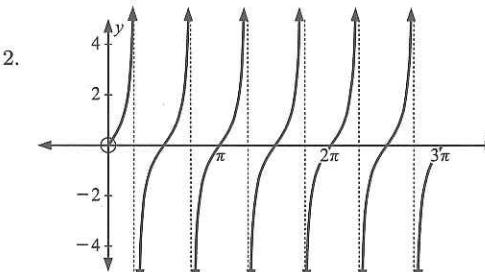
translated $(\frac{\pi}{2}, 0)$.



ii $y = -\tan x$ is $y = \tan x$ reflected in the x-axis.



iii $y = \tan 2x$ comes from $y = \tan x$ under a horizontal compression of factor 2.



2 a translation through $(1, 0)$

b reflection in x-axis

c horizontal stretch, factor 2

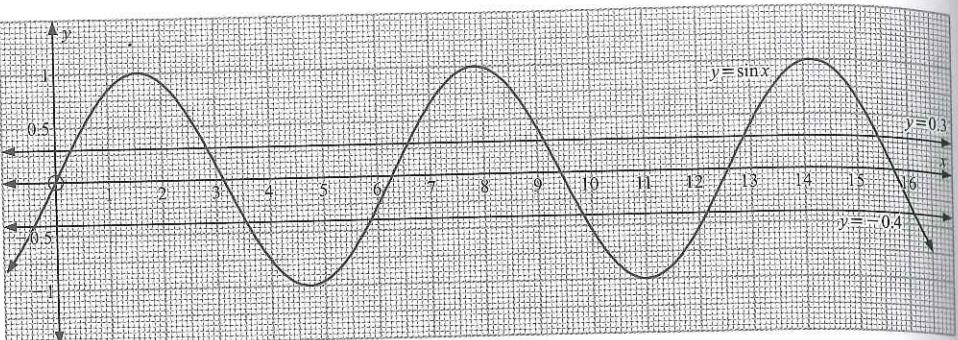
3 a period = $\frac{\pi}{1} = \pi$

b period = $\frac{\pi}{2}$

c period = $\frac{\pi}{n}$

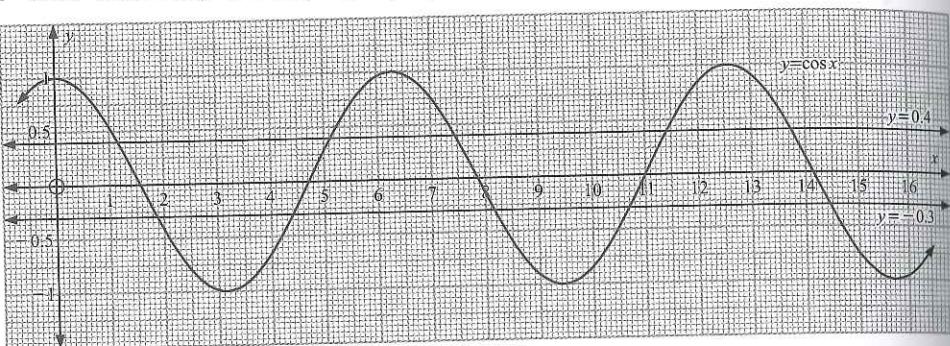
EXERCISE 12F.1

1



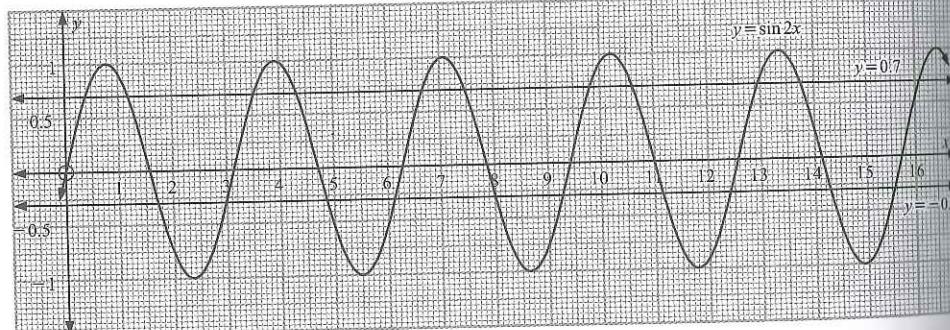
- a When $\sin x = 0.3$, $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$ b When $\sin x = -0.4$, $x \approx 5.9, 9.8, 12.2$

2



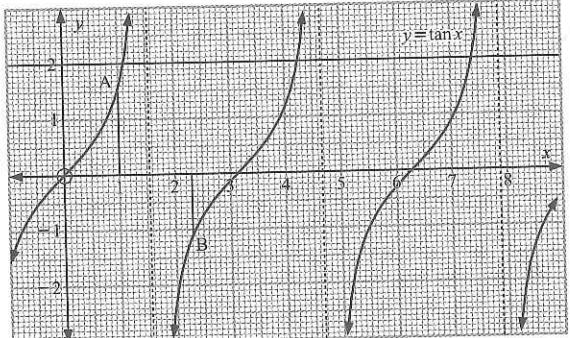
- a When $\cos x = 0.4$, $x \approx 1.2, 5.1, 7.4$ b When $\cos x = -0.3$, $x \approx 4.4, 8.2, 10.7$

3



- a When $\sin(2x) = 0.7$, $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$
b When $\sin(2x) = -0.3$, $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$

4



- a i $\tan 1 \approx 1.6$ {point A}
ii $\tan 2.3 \approx -1.1$ {point B}

b i $\tan 1 \approx 1.557$
ii $\tan 2.3 \approx -1.119$

c i When $\tan x = 2$, $x \approx 1.1, 4.2, 7.4$
ii When $\tan x = -1.4$, $x \approx 2.2, 5.3$

EXERCISE 12F.2

1 Using technology:

- a $\sin(x+2) = 0.0652$ when $x \approx 1.08, 4.35$ b $\sin^2 x + \sin x - 1 = 0$ when $x \approx 0.666, 2.48$
c $x \tan\left(\frac{x^2}{10}\right) = x^2 - 6x + 1$ when $x \approx 0.171, 4.92$
d $2 \sin(2x) \cos x = \ln x$ when $x \approx 1.31, 2.03, 2.85$

$$2 \cos(x-1) + \sin(x+1) = 6x + 5x^2 - x^3 \text{ when } x \approx -0.951, 0.234, 5.98$$

EXERCISE 12F.3

$$1 \quad a \quad x = \frac{\pi}{6} + \frac{k12\pi}{6} \text{ and } 0 \leq x \leq \frac{36\pi}{6}$$

$$\therefore x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$$

$$c \quad x = -\frac{\pi}{2} + \frac{k2\pi}{2} \text{ and } -\frac{8\pi}{2} \leq x \leq \frac{8\pi}{2}$$

$$\therefore x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, -\frac{7\pi}{2}$$

$$2 \quad a \quad \cos x = -\frac{1}{2}, \quad x \in [0, 5\pi]$$

$$\therefore x = -\frac{2\pi}{3} \quad x = -\frac{4\pi}{3} \quad x = -\frac{2\pi}{3} + k2\pi$$

$$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$$

$$e \quad 2 \cos x + \sqrt{3} = 0, \quad 0 \leq x \leq 3\pi$$

$$\therefore \cos x = -\frac{\sqrt{3}}{2}$$

$$\therefore x = -\frac{5\pi}{6} \quad x = -\frac{7\pi}{6} \quad x = -\frac{5\pi}{6} + k2\pi$$

$$\therefore x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$$

$$g \quad 2 \sin(x + \frac{\pi}{3}) = 1, \quad x \in [-3\pi, 3\pi]$$

$$\therefore \sin(x + \frac{\pi}{3}) = \frac{1}{2}$$

$$\therefore x + \frac{\pi}{3} = \frac{\pi}{6} \quad x + \frac{\pi}{3} = \frac{5\pi}{6} \quad x + \frac{\pi}{3} = -\frac{\pi}{2}$$

$$\therefore x = -\frac{2\pi}{3} \quad x = -\frac{7\pi}{6} \quad x = -\frac{13\pi}{6}, -\frac{3\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}, \frac{5\pi}{2}$$

$$b \quad x = -\frac{\pi}{3} + \frac{k6\pi}{3} \text{ and } -\frac{6\pi}{3} \leq x \leq \frac{6\pi}{3}$$

$$\therefore x = -\frac{\pi}{3}, \frac{5\pi}{3}$$

$$d \quad x = \frac{5\pi}{6} + \frac{k3\pi}{6} \text{ and } 0 \leq x \leq \frac{24\pi}{6}$$

$$\therefore x = \frac{2\pi}{6}, \frac{5\pi}{6}, \frac{8\pi}{6}, \frac{11\pi}{6}, \frac{14\pi}{6}, \frac{17\pi}{6}, \frac{20\pi}{6}, \frac{23\pi}{6}$$

$$\therefore x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{7\pi}{3}, \frac{17\pi}{6}, \frac{10\pi}{3}, \frac{23\pi}{6}$$

$$f \quad 2 \sin x - 1 = 0, \quad -2\pi \leq x \leq 2\pi$$

$$\therefore \sin x = \frac{1}{2}$$

$$\therefore x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$h \quad \cos(x - \frac{2\pi}{3}) = \frac{1}{2}, \quad x \in [-2\pi, 2\pi]$$

$$\therefore x - \frac{2\pi}{3} = \frac{\pi}{3} \quad x - \frac{2\pi}{3} = \frac{5\pi}{3}$$

$$\therefore x = \frac{7\pi}{3} \quad x = \frac{5\pi}{3} + k2\pi$$

$$\therefore x = -\frac{5\pi}{3}, -\pi, \frac{\pi}{3}, \pi$$

$$j \quad \sqrt{2} \sin(x - \frac{\pi}{4}) + 1 = 0, \quad x \in [0, 3\pi]$$

$$\therefore \sin(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

$$\therefore x - \frac{\pi}{4} = \frac{3\pi}{2} \quad x - \frac{\pi}{4} = \frac{7\pi}{4}$$

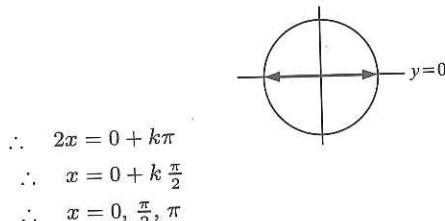
$$\therefore x = \frac{3\pi}{2} + k2\pi \quad x = 0, \frac{3\pi}{2}, 2\pi$$

3 a $X = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} + k\pi$
 $\therefore x - \frac{\pi}{6} = \frac{\pi}{3}$
 $\therefore x = \frac{\pi}{2} + k\pi$

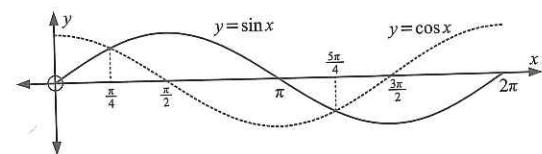
b $\tan 4x = \sqrt{3}$
 $\therefore 4x = \frac{\pi}{3} + k\pi$
 $\therefore x = \frac{\pi}{12} + \frac{k\pi}{4}$

c $\tan^2 x = 3$
 $\therefore \tan x = \pm\sqrt{3}$
 $\therefore x = \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\} + k\pi$

- 4 a The zeros of $y = \sin 2x$ are the solutions of $\sin 2x = 0 \quad \{0 \leq x \leq \pi\}$



5 a



6 a $\sin x = -\cos x, \quad 0 \leq x \leq 2\pi$

$$\begin{aligned} \frac{\sin x}{\cos x} &= \frac{-\cos x}{\cos x} \\ \therefore \tan x &= -1 \\ \therefore x &= \frac{3\pi}{4} + k\pi \\ \therefore x &= \frac{3\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

c $\sin(2x) = \sqrt{3} \cos(2x), \quad 0 \leq x \leq 2\pi$

$$\begin{aligned} \frac{\sin(2x)}{\cos(2x)} &= \sqrt{3} \\ \therefore \tan(2x) &= \sqrt{3} \\ \therefore 2x &= \frac{\pi}{3} + k\pi \\ \therefore x &= \frac{\pi}{6} + \frac{k\pi}{2} \\ \therefore x &= \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, \frac{10\pi}{6} \\ \therefore x &= \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3} \end{aligned}$$

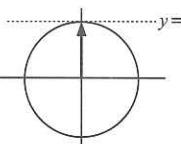
EXERCISE 12G

1 a $P(t) = 7500 + 3000 \sin\left(\frac{\pi t}{8}\right), \quad 0 \leq t \leq 12$

$$\begin{aligned} \text{i} \quad P(0) &= 7500 + 3000 \sin 0 \\ &= 7500 + 0 \\ &= 7500 \text{ grasshoppers} \end{aligned}$$

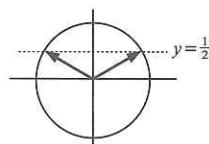
$$\begin{aligned} \text{ii} \quad P(5) &= 7500 + 3000 \sin\left(\frac{5\pi}{8}\right) \\ &\approx 10271.63 \dots \\ &\approx 10300 \text{ grasshoppers} \end{aligned}$$

- b The greatest value of $P(t)$ occurs when $\sin\left(\frac{\pi t}{8}\right) = 1$, so the greatest population is $7500 + 3000 = 10500$ grasshoppers when $\frac{\pi t}{8} = \frac{\pi}{2} + k2\pi$
 $\therefore \frac{t}{8} = \frac{1}{2} + 2k$
 $\therefore t = 4 + 16k$
 $\therefore t = 4 \quad \{\text{as } 0 \leq t \leq 12\}$



So the greatest population occurs after 4 weeks.

c i When $P(t) = 9000$,
 $7500 + 3000 \sin\left(\frac{\pi t}{8}\right) = 9000$
 $\therefore 3000 \sin\left(\frac{\pi t}{8}\right) = 1500$
 $\therefore \sin\left(\frac{\pi t}{8}\right) = \frac{1}{2}$



$$\begin{aligned} \therefore \frac{\pi t}{8} &= \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\} + k2\pi \\ \therefore \frac{t}{8} &= \left\{\frac{1}{6}, \frac{5}{6}\right\} + k2 \\ \therefore t &= \left\{\frac{4}{3}, \frac{20}{3}\right\} + k16 \\ \therefore t &= 1\frac{1}{3} \text{ or } 6\frac{2}{3} \end{aligned}$$

So, the population is 9000 at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks.

- d If $P(t) > 10000$, then

$$\begin{aligned} 7500 + 3000 \sin\left(\frac{\pi t}{8}\right) &> 10000 \\ \therefore 3000 \sin\left(\frac{\pi t}{8}\right) &> 2500 \\ \therefore \sin\left(\frac{\pi t}{8}\right) &> \frac{5}{6} \end{aligned}$$

Solving $\sin\left(\frac{\pi t}{8}\right) = \frac{5}{6}$ using technology

$$t \approx 2.51 \text{ or } 5.49 \quad \text{So, } 2.51 \leq t \leq 5.49 \text{ weeks.}$$

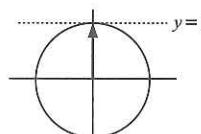
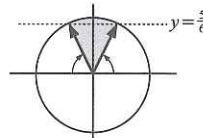
2 $H(t) = 20 - 19 \sin\left(\frac{2\pi t}{3}\right)$

a $H(0) = 20 - 19(0)$
 $= 20 \text{ m}$

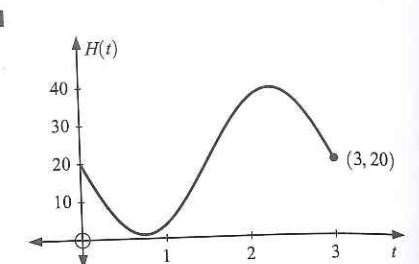
So, at time $t = 0$, the light is 20 m above the ground.

- b H is smallest when $\sin\left(\frac{2\pi t}{3}\right) = 1$

$$\begin{aligned} \frac{2\pi t}{3} &= \frac{\pi}{2} + k2\pi \\ \therefore \frac{2t}{3} &= \frac{1}{2} + k2 \\ \therefore t &= \frac{3}{4} + k3 \\ \therefore t &= \frac{3}{4} \text{ min } \quad \{\text{as } k = 0\} \end{aligned}$$



c period = $\frac{2\pi}{\frac{2\pi}{3}} = 3$ min
 \therefore one revolution takes 3 min



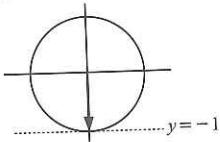
3 $P(t) = 400 + 250 \sin\left(\frac{\pi t}{2}\right)$ years

a $P(0) = 400 + 250(0)$
 $= 400$ water buffalo

c $P(1) = 400 + 250 \sin\left(\frac{\pi}{2}\right)$
 $= 400 + 250 \times 1$
 $= 650$ water buffalo

This is the maximum herd size.

d $P(t)$ is smallest when $\sin\left(\frac{\pi t}{2}\right) = -1$
and is $400 - 250 = 150$ water buffalo.



It occurs when $\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$

$\therefore \frac{t}{2} = \frac{3}{2} + k2$

$\therefore t = 3 + 4k$

So, the first time is after 3 years.

4 a The period is 4 seconds.

$\therefore \frac{2\pi}{B} = 4$

$\therefore B = \frac{\pi}{2}$

Amplitude is 3

$\therefore A = 3$

b X enters the water when $H(t) = 2$

$\therefore 3 \cos\left(\frac{\pi t}{2}\right) + 4 = 2$

$\therefore \cos\left(\frac{\pi t}{2}\right) = -\frac{2}{3}$

Using technology, $t \approx 1.46$ sec

b i $P\left(\frac{1}{2}\right) = 400 + 250 \sin\left(\frac{\pi}{2}\right)$

$= 400 + 250 \sin\left(\frac{\pi}{4}\right)$

$= 400 + 250 \times \frac{1}{\sqrt{2}}$

≈ 577 water buffalo

ii $P(2) = 400 + 250 \sin \pi$

$= 400 + 250(0)$

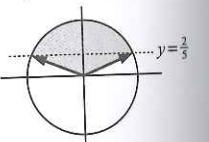
$= 400$ water buffalo

e If $P(t) > 500$ then

$400 + 250 \sin\left(\frac{\pi t}{2}\right) > 500$

$\therefore 250 \sin\left(\frac{\pi t}{2}\right) > 100$

$\therefore \sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$



$\sin\left(\frac{\pi t}{2}\right) = \frac{2}{5}$ when

$\frac{\pi t}{2} = 0.4115$ or $\pi - 0.4115$

$\therefore t \approx 0.262$ or 1.74

So, for $\sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$, $0.26 < t < 1.74$

\therefore the herd first exceeded 500 when $t \approx 0.262$ years.

5 $C(t) = 9.2 \sin \frac{\pi}{7}(t - 4) + 107.8$ cents L⁻¹

a i 107.8 is the median value. Values are between $107.8 - 9.2$ and $107.8 + 9.2$
 $= 98.6$ cents L⁻¹ and 117.0 cents L⁻¹
 \uparrow min. \uparrow max.
 \therefore the statement is true.

ii period = $\frac{2\pi}{\frac{\pi}{7}} = 14$ days \therefore true

b $C(7) = 9.2 \sin \frac{\pi}{7}(3) + 107.8 \approx 116.8$ cents L⁻¹

c When $C(t) = \$1.10$ L⁻¹ then $9.2 \sin \frac{\pi}{7}(t - 4) + 107.8 = 110$

$\therefore \sin \frac{\pi}{7}(t - 4) = \frac{2.2}{9.2} \approx 0.23913$

$\therefore \frac{\pi}{7}(t - 4) \approx 0.23913$ or $\pi - 0.23913$

$\therefore t - 4 \approx 0.533$ or 6.467

$\therefore t \approx 4.53$ or 10.47

So, the price is \$1.10 per litre on the 5th and 11th days.

d The min. cost per litre is $-9.2 + 107.8 = 98.6$ cents L⁻¹

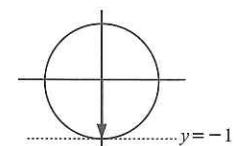
when $\sin \frac{\pi}{7}(t - 4) = -1$ $\therefore 2t - 8 = 21$

$\therefore \frac{\pi}{7}(t - 4) = \frac{3\pi}{2}$ $\therefore 2t = 29$

$\therefore \frac{t - 4}{7} = \frac{3}{2}$ $\therefore t = 14.5 \pm 14k$

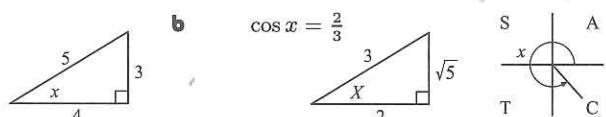
{period is 14 days}

So, the minimum occurred on the 1st day and the 15th day.



EXERCISE 12H

1 a $\sin x = \frac{3}{5}$, $0 \leq x \leq \frac{\pi}{2}$



$\therefore \csc x = \frac{1}{\sin x} = \frac{5}{3}$

$\sec x = \frac{1}{\cos x} = \frac{5}{4}$

$\cot x = \frac{1}{\tan x} = \frac{4}{3}$

$\therefore \sin x = -\frac{\sqrt{5}}{3}$ and $\tan x = -\frac{\sqrt{5}}{2}$

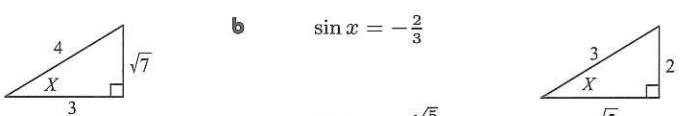
$\therefore \csc x = -\frac{3}{\sqrt{5}}$

$\sec x = \frac{3}{2}$

$\cot x = -\frac{2}{\sqrt{5}}$

b a $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ b $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$ c $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ d $\tan(\pi) = 0$
 $\therefore \csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}$ $\therefore \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}$ $\therefore \sec\left(\frac{5\pi}{6}\right) = -\frac{2}{\sqrt{3}}$
 $\therefore \cot(\pi)$ is undefined.

2 a $\cos x = \frac{3}{4}$



$\therefore \sin x = -\frac{\sqrt{7}}{4}$

$\tan x = -\frac{\sqrt{7}}{3}$

$\csc x = -\frac{4}{\sqrt{7}}$

$\sec x = \frac{4}{3}$

$\cot x = -\frac{3}{\sqrt{7}}$

$\therefore \cos x = -\frac{\sqrt{5}}{3}$

$\tan x = \frac{2}{\sqrt{5}}$

$\csc x = -\frac{3}{2}$

$\sec x = -\frac{3}{\sqrt{5}}$

$\cot x = \frac{\sqrt{5}}{2}$

k

$$\begin{aligned} & \frac{1 + \cot \theta}{\csc \theta} - \frac{\sec \theta}{\tan \theta + \cot \theta} \\ &= \sin \theta \left(1 + \frac{\cos \theta}{\sin \theta}\right) - \frac{1}{\cos \theta \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}\right)} \\ &= \sin \theta + \cos \theta - \frac{1}{\sin \theta + \frac{\cos^2 \theta}{\sin \theta}} \\ &= \sin \theta + \cos \theta - \frac{\sin \theta}{\sin^2 \theta + \cos^2 \theta} \\ &= \sin \theta + \cos \theta - \sin \theta \\ &= \cos \theta \end{aligned}$$

3 a

$$\begin{aligned} \sec A - \cos A &= \frac{1}{\cos A} - \cos A \\ &= \frac{1 - \cos^2 A}{\cos A} \\ &= \frac{\sin^2 A}{\cos A} \\ &= \frac{\sin A}{\cos A} \times \sin A \\ &= \tan A \sin A \end{aligned}$$

c

$$\begin{aligned} & \frac{\cos \alpha}{1 - \tan \alpha} + \frac{\sin \alpha}{1 - \cot \alpha} \\ &= \frac{\cos \alpha}{1 - \frac{\sin \alpha}{\cos \alpha}} + \frac{\sin \alpha}{1 - \frac{\cos \alpha}{\sin \alpha}} \\ &= \frac{\cos \alpha}{\frac{\cos \alpha - \sin \alpha}{\cos \alpha}} + \frac{\sin \alpha}{\frac{\sin \alpha - \cos \alpha}{\sin \alpha}} \\ &= \frac{\cos^2 \alpha}{\cos \alpha - \sin \alpha} + \frac{\sin^2 \alpha}{\sin \alpha - \cos \alpha} \\ &= \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos \alpha - \sin \alpha} \\ &= \frac{(\cos \alpha + \sin \alpha)(\cos \alpha - \sin \alpha)}{\cos \alpha - \sin \alpha} \\ &= \sin \alpha + \cos \alpha \end{aligned}$$

EXERCISE 12J

1 a

$$\begin{aligned} \sin(90^\circ + \theta) &= \sin 90^\circ \cos \theta + \cos 90^\circ \sin \theta \\ &= (1) \cos \theta + (0) \sin \theta \\ &= \cos \theta \end{aligned}$$

c

$$\begin{aligned} \sin(180^\circ - \alpha) &= \sin 180^\circ \cos \alpha - \cos 180^\circ \sin \alpha \\ &= (0) \cos \alpha - (-1) \sin \alpha \\ &= \sin \alpha \end{aligned}$$

l

$$\begin{aligned} & \frac{\cos^2 \beta - \sin^2 \beta}{\cos \beta - \sin \beta} \\ &= \frac{(\cos \beta + \sin \beta)(\cos \beta - \sin \beta)}{\cancel{\cos \beta - \sin \beta}} \\ &= \cos \beta + \sin \beta \end{aligned}$$

m

$$\begin{aligned} \frac{\tan^2 \theta}{\sec \theta - 1} &= \frac{\sec^2 \theta - 1}{\sec \theta - 1} \\ &= \frac{(\sec \theta + 1)(\sec \theta - 1)}{\cancel{\sec \theta - 1}} \\ &= \sec \theta + 1 \end{aligned}$$

b

$$\begin{aligned} \frac{\cos \theta}{1 - \sin \theta} &= \frac{\cos \theta(1 + \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} \\ &= \frac{\cos \theta + \cos \theta \sin \theta}{1 - \sin^2 \theta} \\ &= \frac{\cos \theta + \cos \theta \sin \theta}{\cos^2 \theta} \\ &= \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} \\ &= \sec \theta + \tan \theta \end{aligned}$$

d

$$\begin{aligned} \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} &= \frac{\sin^2 \theta + (1 + \cos \theta)^2}{(1 + \cos \theta) \sin \theta} \\ &= \frac{\sin^2 \theta + 1 + 2 \cos \theta + \cos^2 \theta}{(1 + \cos \theta) \sin \theta} \\ &= \frac{2 + 2 \cos \theta}{(1 + \cos \theta) \sin \theta} \\ &= \frac{2(1 + \cos \theta)}{(1 + \cos \theta) \sin \theta} \\ &= \frac{2}{\sin \theta} \\ &= 2 \csc \theta \end{aligned}$$

b

$$\begin{aligned} \cos(90^\circ + \theta) &= \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta \\ &= (0) \cos \theta - (1) \sin \theta \\ &= -\sin \theta \end{aligned}$$

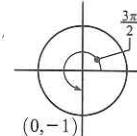
d

$$\begin{aligned} \cos(\pi + \alpha) &= \cos \pi \cos \alpha - \sin \pi \sin \alpha \\ &= (-1) \cos \alpha - (0) \sin \alpha \\ &= -\cos \alpha \end{aligned}$$

e

$$\begin{aligned} \sin(2\pi - A) &= \sin 2\pi \cos A - \cos 2\pi \sin A \\ &= (0) \cos A - (1) \sin A \\ &= -\sin A \end{aligned}$$

f

$$\begin{aligned} \cos\left(\frac{3\pi}{2} - \theta\right) &= \cos\left(\frac{3\pi}{2}\right) \cos \theta + \sin\left(\frac{3\pi}{2}\right) \sin \theta \\ &= (0) \cos \theta + (-1) \sin \theta \\ &= -\sin \theta \end{aligned}$$


g

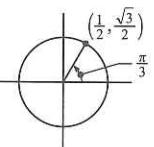
$$\begin{aligned} \tan\left(\frac{\pi}{4} + \theta\right) &= \frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \\ &= \frac{1 + \tan \theta}{1 - \tan \theta} \end{aligned}$$

h

$$\begin{aligned} \tan\left(\theta - \frac{3\pi}{4}\right) &= \frac{\tan \theta - \tan \frac{3\pi}{4}}{1 + \tan \theta \tan \frac{3\pi}{4}} \\ &= \frac{\tan \theta - (-1)}{1 + \tan \theta(-1)} \\ &= \frac{1 + \tan \theta}{1 - \tan \theta} \\ &= \tan \theta \end{aligned}$$

i

$$\begin{aligned} \sin\left(\theta + \frac{\pi}{3}\right) &= \sin \theta \cos\left(\frac{\pi}{3}\right) + \cos \theta \sin\left(\frac{\pi}{3}\right) \\ &= \sin \theta \times \left(\frac{1}{2}\right) + \cos \theta \times \left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta \end{aligned}$$

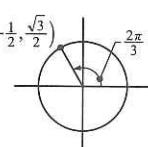


j

$$\begin{aligned} \cos\left(\theta + \frac{\pi}{4}\right) &= \cos \theta \cos\left(\frac{\pi}{4}\right) - \sin \theta \sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta \\ &= -\frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \end{aligned}$$

b

$$\begin{aligned} \cos\left(\frac{2\pi}{3} - \theta\right) &= \cos\left(\frac{2\pi}{3}\right) \cos \theta + \sin\left(\frac{2\pi}{3}\right) \sin \theta \\ &= \left(-\frac{1}{2}\right) \cos \theta + \left(\frac{\sqrt{3}}{2}\right) \sin \theta \\ &= -\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta \end{aligned}$$



3 a

$$\begin{aligned} \cos 2\theta \cos \theta + \sin 2\theta \sin \theta &= \cos(2\theta - \theta) \\ &= \cos \theta \end{aligned}$$

c

$$\begin{aligned} \cos A \sin B - \sin A \cos B &= \sin B \cos A - \cos B \sin A \\ &= \sin(B - A) \end{aligned}$$

e

$$\begin{aligned} \sin \phi \sin \theta - \cos \phi \cos \theta &= -[\cos \phi \cos \theta - \sin \phi \sin \theta] \\ &= -\cos(\phi + \theta) \end{aligned}$$

g

$$\begin{aligned} \frac{\tan 2\theta - \tan \theta}{1 + \tan 2\theta \tan \theta} &= \tan(2\theta - \theta) \\ &= \tan \theta \end{aligned}$$

4 a

$$\begin{aligned} \frac{\sin 2\theta}{1 + \cos 2\theta} - \tan \theta &= \frac{\sin 2\theta}{1 + \cos 2\theta} - \frac{\sin \theta}{\cos \theta} \\ &= \frac{\sin 2\theta \cos \theta - \sin \theta(1 + \cos 2\theta)}{(1 + \cos 2\theta) \cos \theta} \\ &= \frac{\sin 2\theta \cos \theta - \sin \theta - \sin \theta \cos 2\theta}{(1 + \cos 2\theta) \cos \theta} \\ &= 0 \end{aligned}$$

b

$$\begin{aligned} \sin 2A \cos A + \cos 2A \sin A &= \sin(2A + A) \\ &= \sin 3A \end{aligned}$$

d

$$\begin{aligned} \sin \alpha \sin \beta + \cos \alpha \cos \beta &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ &= \cos(\alpha - \beta) \end{aligned}$$

f

$$\begin{aligned} 2 \sin \alpha \cos \beta - 2 \cos \alpha \sin \beta &= 2 [\sin \alpha \cos \beta - \cos \alpha \sin \beta] \\ &= 2 \sin(\alpha - \beta) \end{aligned}$$

h

$$\begin{aligned} \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} &= \tan(2A + A) \\ &= \tan 3A \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \tan \theta + \cot 2\theta &= \frac{\sin \theta}{\cos \theta} + \frac{\cos 2\theta}{\sin 2\theta} \\ &= \frac{\sin \theta \sin 2\theta + \cos 2\theta \cos \theta}{\cos \theta \sin 2\theta} \\ &= \frac{\cos 2\theta \cos \theta + \sin 2\theta \sin \theta}{\cos \theta \sin 2\theta} \\ &= \frac{\cos(2\theta - \theta)}{\cos \theta \sin 2\theta} \\ &= \csc 2\theta \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad &\cos(\alpha + \beta) \cos(\alpha - \beta) - \sin(\alpha + \beta) \sin(\alpha - \beta) \\ &= \cos[(\alpha + \beta) + (\alpha - \beta)] \\ &= \cos 2\alpha \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\sin(\theta - 2\phi) \cos(\theta + \phi) - \cos(\theta - 2\phi) \sin(\theta + \phi) \\ &= \sin[(\theta - 2\phi) - (\theta + \phi)] \\ &= \sin(-3\phi) \\ &= -\sin 3\phi \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad &\cos 75^\circ \\ &= \cos(45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \\ &= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad &\cos\left(\frac{13\pi}{12}\right) \\ &= \cos\left(\frac{10\pi}{12} + \frac{3\pi}{12}\right) \\ &= \cos\left(\frac{5\pi}{6} + \frac{\pi}{4}\right) \\ &= \cos\left(\frac{5\pi}{6}\right) \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{5\pi}{6}\right) \sin\left(\frac{\pi}{4}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad &\tan\left(\frac{5\pi}{12}\right) \\ &= \tan\left(\frac{5 \times 180^\circ}{12}\right) \\ &= \tan 75^\circ \\ &= \tan(45^\circ + 30^\circ) \\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1)\left(\frac{1}{\sqrt{3}}\right)} \end{aligned}$$

$$\begin{aligned} &= \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) \\ &= \frac{3+\sqrt{3}+\sqrt{3}+1}{3-1} \\ &= \frac{4+2\sqrt{3}}{2} \\ &= 2+\sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad &\frac{\sin 2\theta}{\sin \theta} - \frac{\cos 2\theta}{\cos \theta} = \frac{\sin 2\theta \cos \theta - \sin \theta \cos 2\theta}{\sin \theta \cos \theta} \\ &= \frac{\sin(2\theta - \theta)}{\sin \theta \cos \theta} \\ &= \frac{\sin \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\cos \theta} \\ &= \sec \theta \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad &\cos \alpha \cos(\beta - \alpha) \\ &\quad - \sin \alpha \sin(\beta - \alpha) \\ &= \cos[\alpha + (\beta - \alpha)] \\ &= \cos \beta \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\sin 105^\circ \\ &= \sin(60^\circ + 45^\circ) \\ &= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\ &= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\ &= \left(\frac{-\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{-\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

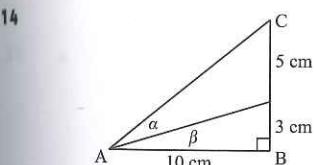
$$\begin{aligned} \mathbf{b} \quad &\tan 105^\circ \\ &= \tan(180^\circ - 75^\circ) \\ &= \frac{\tan 180^\circ - \tan 75^\circ}{1 + \tan 180^\circ \tan 75^\circ} \\ &= \frac{0 - (2+\sqrt{3})}{1 + (0)(2+\sqrt{3})} \quad \{ \text{using } \mathbf{1} \} \\ &= -2 - \sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{8} \quad \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{\frac{2}{3} - \frac{1}{5}}{1 - \left(\frac{2}{3}\right)\left(-\frac{1}{5}\right)} \\ &= \frac{\frac{10}{15} - \frac{3}{15}}{1 + \frac{2}{15}} = \frac{\frac{7}{15}}{\frac{17}{15}} = \frac{7}{17} \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad \mathbf{a} \quad &\tan(A + \frac{\pi}{4}) \tan(A - \frac{\pi}{4}) \\ &= \frac{\tan A + \tan \frac{\pi}{4}}{1 - \tan A \tan \frac{\pi}{4}} \times \frac{\tan A - \tan \frac{\pi}{4}}{1 + \tan A \tan \frac{\pi}{4}} \\ &= \left(\frac{\tan A + 1}{1 - \tan A}\right) \left(\frac{\tan A - 1}{1 + \tan A}\right) \\ &= \frac{\tan^2 A - 1}{1 - \tan^2 A} \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{11} \quad &\frac{\tan 80^\circ - \tan 20^\circ}{1 + \tan 80^\circ \tan 20^\circ} \\ &= \tan(80^\circ - 20^\circ) \\ &= \tan 60^\circ = \sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{13} \quad &\tan(A-B) \tan(A+B) = 1 \\ \therefore &\left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right) \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) = 1 \\ \therefore &\frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} = 1 \end{aligned}$$



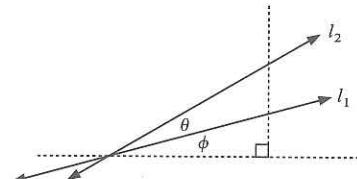
$$\begin{aligned} \tan \beta &= \frac{3}{10} \\ \tan(\alpha + \beta) &= \frac{8}{10} \\ \therefore \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} &= \frac{8}{10} \\ \therefore \frac{\tan \alpha + \frac{3}{10}}{1 - \tan \alpha \left(\frac{3}{10}\right)} &= \frac{4}{5} \\ \therefore 5 \tan \alpha + \frac{15}{10} &= 4 - \frac{12}{10} \tan \alpha \\ \therefore \frac{31}{5} \tan \alpha &= \frac{5}{2} \\ \therefore \tan \alpha &= \frac{25}{62} \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad \tan(A + \frac{\pi}{4}) &= \frac{\tan A + \tan \frac{\pi}{4}}{1 - \tan A \tan \frac{\pi}{4}} \\ &= \frac{\frac{3}{4} + 1}{1 - (\frac{3}{4})(1)} \\ &= \frac{\frac{7}{4}}{\frac{1}{4}} = 7 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B) \tan(A-B)} \\ &= \tan[(A+B) + (A-B)] \\ &= \tan 2A \end{aligned}$$

$$\begin{aligned} \mathbf{12} \quad &\tan(A+B) = \frac{3}{5} \\ \therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} &= \frac{3}{5} \\ \therefore \frac{\tan A + \frac{2}{3}}{1 - \tan A (\frac{2}{3})} &= \frac{3}{5} \\ \therefore 5 \tan A + \frac{10}{3} &= 3 - 2 \tan A \\ \therefore 7 \tan A &= -\frac{1}{3} \\ \therefore \tan A &= -\frac{1}{21} \end{aligned}$$

$$\begin{aligned} \therefore \tan^2 A - \tan^2 B &= 1 - \tan^2 A \tan^2 B \\ \therefore \tan^2 A (\tan^2 B + 1) &= 1 + \tan^2 B \\ \therefore \tan^2 A &= 1 \\ \therefore \tan A &= \pm 1 \end{aligned}$$



θ is the acute angle between the lines l_1 and l_2 .

$$\begin{aligned} \tan \phi &= \frac{1}{2} \quad \{ \text{gradient of } l_1 \} \\ \tan(\theta + \phi) &= \frac{2}{3} \quad \{ \text{gradient of } l_2 \} \end{aligned}$$

$$\begin{aligned} \therefore \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} &= \frac{2}{3} \\ \therefore \frac{\tan \theta + \frac{1}{2}}{1 - \tan \theta \left(\frac{1}{2}\right)} &= \frac{2}{3} \\ \therefore 3 \tan \theta + \frac{3}{2} &= 2 - \tan \theta \\ \therefore 4 \tan \theta &= \frac{1}{2} \\ \therefore \tan \theta &= \frac{1}{8} \\ \therefore \text{the tangent of the acute angle is } \frac{1}{8}. \end{aligned}$$

16 $\tan(A + B + C) = \tan[(A + B) + C]$

$$\begin{aligned} &= \frac{\tan(A + B) + \tan C}{1 - \tan(A + B)\tan C} \\ &= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan C} \\ &= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B) \times \tan C} \\ &= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C} \end{aligned}$$

If A, B and C are the angles of a triangle then $A + B + C = 180^\circ$

$$\therefore \tan(A + B + C) = 0$$

$$\begin{aligned} \therefore \tan A + \tan B + \tan C - \tan A \tan B \tan C &= 0 \\ \therefore \tan A + \tan B + \tan C &= \tan A \tan B \tan C \end{aligned}$$

17 a $\sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right)$

$$\begin{aligned} &= \sqrt{2} [\cos \theta \cos\left(\frac{\pi}{4}\right) - \sin \theta \sin\left(\frac{\pi}{4}\right)] \\ &= \sqrt{2} \left[\frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta \right] \\ &= \cos \theta - \sin \theta \end{aligned}$$

c $\cos(\alpha + \beta) - \cos(\alpha - \beta)$

$$\begin{aligned} &= \cos \alpha \cos \beta - \sin \alpha \sin \beta - [\cos \alpha \cos \beta + \sin \alpha \sin \beta] \\ &= \cancel{\cos \alpha \cos \beta} - \sin \alpha \sin \beta - \cancel{\cos \alpha \cos \beta} - \sin \alpha \sin \beta \\ &= -2 \sin \alpha \sin \beta \end{aligned}$$

d $\cos(\alpha + \beta) \cos(\alpha - \beta)$

$$\begin{aligned} &= [\cos \alpha \cos \beta - \sin \alpha \sin \beta] [\cos \alpha \cos \beta + \sin \alpha \sin \beta] \\ &= \cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta \\ &= \cos^2 \alpha [1 - \sin^2 \beta] - [1 - \cos^2 \alpha] \sin^2 \beta \\ &= \cos^2 \alpha - \cancel{\cos^2 \alpha \sin^2 \beta} - \sin^2 \beta + \cancel{\cos^2 \alpha \sin^2 \beta} \\ &= \cos^2 \alpha - \sin^2 \beta \end{aligned}$$

18 $\tan \alpha = \frac{1}{3}$ and $\tan \beta = \frac{1}{2}$

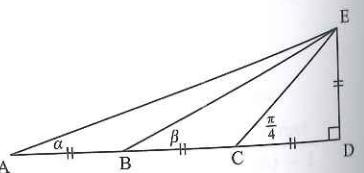
$$\begin{aligned} \therefore \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ &= \frac{\frac{1}{3} + \frac{1}{2}}{1 - (\frac{1}{3})(\frac{1}{2})} \\ &= \frac{\frac{5}{6}}{\frac{5}{6}} = 1 \\ \therefore \alpha + \beta &= \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

But clearly both α and $\beta < \frac{\pi}{2}$

$$\therefore \alpha + \beta = \frac{\pi}{4}$$

19 a $\sin(A + B) + \sin(A - B)$

$$\begin{aligned} &= \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B \\ &= 2 \sin A \cos B \end{aligned}$$



b Using a, $\sin A \cos B = \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B)$

i $\sin 3\theta \cos \theta$
 $= \frac{1}{2} \sin(3\theta + \theta) + \frac{1}{2} \sin(3\theta - \theta)$
 $= \frac{1}{2} \sin 4\theta + \frac{1}{2} \sin 2\theta$

ii $\sin 6\alpha \cos \alpha$
 $= \frac{1}{2} \sin(6\alpha + \alpha) + \frac{1}{2} \sin(6\alpha - \alpha)$
 $= \frac{1}{2} \sin 7\alpha + \frac{1}{2} \sin 5\alpha$

iii $2 \sin 5\beta \cos \beta$
 $= 2 [\frac{1}{2} \sin(5\beta + \beta) + \frac{1}{2} \sin(5\beta - \beta)]$
 $= \sin 6\beta + \sin 4\beta$

iv $4 \cos \theta \sin 4\theta$
 $= 4 [\sin 4\theta \cos \theta]$
 $= 4 [\frac{1}{2} \sin 5\theta + \frac{1}{2} \sin 3\theta]$
 $= 2 \sin 5\theta + 2 \sin 3\theta$

v $6 \cos 4\alpha \sin 3\alpha$
 $= 6 \sin 3\alpha \cos 4\alpha$
 $= 6 [\frac{1}{2} \sin 7\alpha + \frac{1}{2} \sin(-\alpha)]$
 $= 3 \sin 7\alpha + 3 \sin(-\alpha)$
 $= 3 \sin 7\alpha - 3 \sin \alpha$

vi $\frac{1}{3} \cos 5A \sin 3A$
 $= \frac{1}{3} \sin 3A \cos 5A$
 $= \frac{1}{3} [\frac{1}{2} \sin 8A + \frac{1}{2} \sin(-2A)]$
 $= \frac{1}{6} \sin 8A - \frac{1}{6} \sin 2A$

20 a $\cos(A + B) + \cos(A - B)$

$$\begin{aligned} &= \cos A \cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B} \\ &= 2 \cos A \cos B \end{aligned}$$

b $\therefore \cos A \cos B = \frac{1}{2} \cos(A + B) + \frac{1}{2} \cos(A - B)$

i $\cos 4\theta \cos \theta$
 $= \frac{1}{2} \cos 5\theta + \frac{1}{2} \cos 3\theta$

ii $\cos 7\alpha \cos \alpha$
 $= \frac{1}{2} \cos 8\alpha + \frac{1}{2} \cos 6\alpha$

iii $2 \cos 3\beta \cos \beta$
 $= 2 [\frac{1}{2} \cos 4\beta + \frac{1}{2} \cos 2\beta]$
 $= \cos 4\beta + \cos 2\beta$

iv $6 \cos x \cos 7x$

$$\begin{aligned} &= 6 \cos 7x \cos x \\ &= 6 [\frac{1}{2} \cos 8x + \frac{1}{2} \cos 6x] \\ &= 3 \cos 8x + 3 \cos 6x \end{aligned}$$

v $3 \cos P \cos 4P$

$$\begin{aligned} &= 3 \cos 4P \cos P \\ &= 3 [\frac{1}{2} \cos 5P + \frac{1}{2} \cos 3P] \\ &= \frac{3}{2} \cos 5P + \frac{3}{2} \cos 3P \end{aligned}$$

vi $\frac{1}{4} \cos 4x \cos 2x$
 $= \frac{1}{4} [\frac{1}{2} \cos 6x + \frac{1}{2} \cos 2x]$
 $= \frac{1}{8} \cos 6x + \frac{1}{8} \cos 2x$

21 a $\cos(A - B) - \cos(A + B)$

$$\begin{aligned} &= \cancel{\cos A \cos B} + \sin A \sin B - [\cancel{\cos A \cos B} - \sin A \sin B] \\ &= \sin A \sin B + \sin A \sin B \\ &= 2 \sin A \sin B \end{aligned}$$

b $\therefore \sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B)$

i $\sin 3\theta \sin \theta$
 $= \frac{1}{2} \cos 2\theta - \frac{1}{2} \cos 4\theta$

ii $\sin 6\alpha \sin \alpha$
 $= \frac{1}{2} \cos 5\alpha - \frac{1}{2} \cos 7\alpha$

iii $2 \sin 5\beta \sin \beta$
 $= 2 [\frac{1}{2} \cos 4\beta - \frac{1}{2} \cos 6\beta]$
 $= \cos 4\beta - \cos 6\beta$

iv $4 \sin \theta \sin 4\theta$

$$\begin{aligned} &= 4 \sin 4\theta \sin \theta \\ &= 4 [\frac{1}{2} \cos 3\theta - \frac{1}{2} \cos 5\theta] \\ &= 2 \cos 3\theta - 2 \cos 5\theta \end{aligned}$$

v $10 \sin 2A \sin 8A$

$$\begin{aligned} &= 10 \sin 8A \sin 2A \\ &= 10 [\frac{1}{2} \cos 6A - \frac{1}{2} \cos 10A] \\ &= 5 \cos 6A - 5 \cos 10A \\ &= \frac{1}{10} \cos 4M - \frac{1}{10} \cos 10M \end{aligned}$$

22 (1) becomes $\sin A \cos A = \frac{1}{2} \sin 2A$

(2) becomes $\cos^2 A = \frac{1}{2} \cos 2A + \frac{1}{2} \cos 0, \quad \therefore \cos^2 A = \frac{1}{2} \cos 2A + \frac{1}{2}$

(3) becomes $\sin^2 A = \frac{1}{2} \cos 0 - \frac{1}{2} \cos 2A, \quad \therefore \sin^2 A = \frac{1}{2} - \frac{1}{2} \cos 2A$

- 23** **a** $A + B = S \quad \therefore 2A = S + D \quad \therefore A = \frac{S+D}{2}$
 $A - B = D$ and $B = S - A = S - \left(\frac{S+D}{2}\right) = \frac{2S}{2} - \left(\frac{S+D}{2}\right) = \frac{2S-S-D}{2} = \frac{S-D}{2}$
- b** $\sin A \cos B = \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B)$ becomes $\sin\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) = \frac{1}{2} \sin S + \frac{1}{2} \sin D$
or $\sin S + \sin D = 2 \sin\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) \dots (4)$
- c** Replacing D by $(-D)$ in (4) gives
 $\sin S + \sin(-D) = 2 \sin\left(\frac{S-D}{2}\right) \cos\left(\frac{S+D}{2}\right)$
or $\sin S - \sin D = 2 \cos\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)$
- d** $\cos A \cos B = \frac{1}{2} \cos(A+B) + \frac{1}{2} \cos(A-B)$ becomes
 $\cos\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) = \frac{1}{2} \cos S + \frac{1}{2} \cos D$
or $\cos S + \cos D = 2 \cos\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right)$
- e** $\sin A \sin B = \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B)$ becomes
 $\sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right) = \frac{1}{2} \cos D - \frac{1}{2} \cos S$
or $\cos D - \cos S = 2 \sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)$
or $\cos S - \cos D = -2 \sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)$

- 24** **a** $\sin 5x + \sin x$
 $= 2 \sin\left(\frac{5x+x}{2}\right) \cos\left(\frac{5x-x}{2}\right)$
 $= 2 \sin 3x \cos 2x$
- c** $\cos 3\alpha - \cos \alpha$
 $= -2 \sin\left(\frac{3\alpha+\alpha}{2}\right) \sin\left(\frac{3\alpha-\alpha}{2}\right)$
 $= -2 \sin 2\alpha \sin \alpha$
- e** $\cos 7\alpha - \cos \alpha$
 $= -2 \sin\left(\frac{7\alpha+\alpha}{2}\right) \sin\left(\frac{7\alpha-\alpha}{2}\right)$
 $= -2 \sin 4\alpha \sin 3\alpha$
- g** $\cos 2B - \cos 4B$
 $= -[\cos 4B - \cos 2B]$
 $= -2 \sin\left(\frac{4B+2B}{2}\right) \sin\left(\frac{4B-2B}{2}\right)$
 $= 2 \sin 3B \sin B$
- i** $\cos(x+h) - \cos x$
 $= -2 \sin\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right)$
 $= -2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)$
 $= -2 \sin\left(x+\frac{h}{2}\right) \sin\left(\frac{h}{2}\right)$

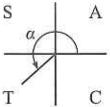
EXERCISE 12K

1 **a** $\sin 2A = 2 \sin A \cos A$
 $= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$
 $= \frac{24}{25}$

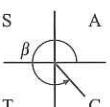
b $\cos 2A = \cos^2 A - \sin^2 A$
 $= \frac{9}{25} - \frac{16}{25}$
 $= -\frac{7}{25}$

2 **a** $\cos 2A = 2 \cos^2 A - 1$
 $= 2\left(\frac{1}{3}\right)^2 - 1$
 $= 2 \times \frac{1}{9} - 1$
 $= \frac{2}{9} - 1$
 $= -\frac{7}{9}$

3 **a** $\sin \alpha = -\frac{2}{3}$
 α is in Q3
 $\therefore \cos \alpha < 0$



b $\cos \beta = \frac{2}{5}$
 β is in Q4
 $\therefore \sin \beta < 0$



4 α is acute $\therefore \cos \alpha$ and $\sin \alpha$ are positive

a $\cos 2\alpha = 2 \cos^2 \alpha - 1$
 $\therefore -\frac{7}{9} = 2 \cos^2 \alpha - 1$
 $\therefore 2 \cos^2 \alpha = \frac{2}{9}$
 $\therefore \cos^2 \alpha = \frac{1}{9}$
 $\therefore \cos \alpha = \frac{1}{3}$

5 $\tan 2A = \frac{21}{20}$
 $\therefore \frac{2 \tan A}{1 - \tan^2 A} = \frac{21}{20}$
 $\therefore 40 \tan A = 21 - 21 \tan^2 A$
 $\therefore 21 \tan^2 A + 40 \tan A - 21 = 0$
 $\therefore (7 \tan A - 3)(3 \tan A + 7) = 0$
 $\therefore \tan A = \frac{3}{7}$ or $-\frac{7}{3}$

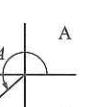
but A is obtuse $\therefore \tan A$ is negative
 $\therefore \tan A = -\frac{7}{3}$

7 $\tan\left(\frac{\pi}{4}\right) = 1$
 $\therefore \tan(2 \times \frac{\pi}{8}) = 1$
 $\therefore \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2(\frac{\pi}{8})} = 1$
 $\therefore 2 \tan(\frac{\pi}{8}) = 1 - \tan^2(\frac{\pi}{8})$
 $\therefore \tan^2(\frac{\pi}{8}) + 2 \tan(\frac{\pi}{8}) - 1 = 0$

$\therefore \tan(\frac{\pi}{8}) = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2}$
 $= \frac{-2 \pm 2\sqrt{2}}{2}$
 $= -1 \pm \sqrt{2}$

but $\frac{\pi}{8}$ is in Q1 $\therefore \tan(\frac{\pi}{8})$ is positive
 $\therefore \tan(\frac{\pi}{8}) = \sqrt{2} - 1$

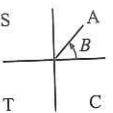
8 $\sin A = -\frac{1}{3}$
 A is in Q3
 $\therefore \cos A < 0$



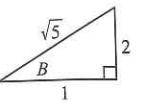
$\cos^2 A + \sin^2 A = 1$
 $\therefore \cos^2 A + \frac{1}{9} = 1$
 $\therefore \cos^2 A = \frac{8}{9}$
 $\therefore \cos A = -\frac{2\sqrt{2}}{3}$

$\therefore \tan A = \frac{\sin A}{\cos A}$
 $= \frac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}}$
 $= \frac{1}{2\sqrt{2}}$

$$\begin{aligned}\cos B &= \frac{1}{\sqrt{5}} \\ B \text{ is in Q1} \quad \therefore \sin B &> 0\end{aligned}$$



$$\begin{aligned}\therefore \sin B &= \frac{2}{\sqrt{5}} \\ \tan B &= 2\end{aligned}$$



$$\begin{aligned}\text{a} \quad &\tan(A+B) \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{\frac{1}{2\sqrt{2}} + 2}{1 - (\frac{1}{2\sqrt{2}})(2)} \times \left(\frac{2\sqrt{2}}{2\sqrt{2}}\right) \\ &= \frac{1+4\sqrt{2}}{2\sqrt{2}-2} \times \left(\frac{2\sqrt{2}+2}{2\sqrt{2}+2}\right) \\ &= \frac{2\sqrt{2}+2+16+8\sqrt{2}}{8-4} \\ &= \frac{18+10\sqrt{2}}{4} = \frac{9+5\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\text{9} \quad &\left[\cos\left(-\frac{\pi}{12}\right) + \sin\left(-\frac{\pi}{12}\right)\right]^2 \\ &= \cos^2\left(-\frac{\pi}{12}\right) + 2\cos\left(-\frac{\pi}{12}\right)\sin\left(-\frac{\pi}{12}\right) + \sin^2\left(-\frac{\pi}{12}\right) \\ &= 1 + 2\cos\left(-\frac{\pi}{12}\right)\sin\left(-\frac{\pi}{12}\right) \\ &= 1 + \sin\left(\frac{\pi}{6}\right) \quad \{ \sin 2A = 2\cos A \sin A \} \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2}\end{aligned}$$

$$\text{10} \quad \text{a} \quad 2\sin\alpha\cos\alpha = \sin 2\alpha$$

$$\text{d} \quad 2\cos^2\beta - 1 = \cos 2\beta$$

$$\text{g} \quad 2\sin^2 M - 1 = -(1 - 2\sin^2 M) = -\cos 2M$$

$$\text{j} \quad 2\sin 2A \cos 2A = \sin 2(2A) = \sin 4A$$

$$\text{m} \quad 1 - 2\cos^2 3\beta = -(2\cos^2 3\beta - 1) = -\cos 2(3\beta) = -\cos 6\beta$$

$$\text{p} \quad \cos^2 2A - \sin^2 2A = \cos 2(2A) = \cos 4A$$

$$\text{b} \quad 4\cos\alpha\sin\alpha = 2(2\sin\alpha\cos\alpha) = 2\sin 2\alpha$$

$$\text{e} \quad 1 - 2\cos^2\phi = -(2\cos^2\phi - 1) = -\cos 2\phi$$

$$\text{h} \quad \cos^2\alpha - \sin^2\alpha = \cos 2\alpha$$

$$\text{k} \quad 2\cos 3\alpha \sin 3\alpha = \sin 2(3\alpha) = \sin 6\alpha$$

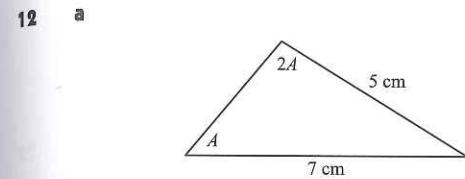
$$\text{n} \quad 1 - 2\sin^2 5\alpha = \cos 2(5\alpha) = \cos 10\alpha$$

$$\text{q} \quad \cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right) = \cos 2\left(\frac{\alpha}{2}\right) = \cos\alpha$$

$$\begin{aligned}\text{b} \quad \tan 2A &= \frac{2\tan A}{1 - \tan^2 A} \\ &= \frac{2\left(\frac{1}{2\sqrt{2}}\right)}{1 - \left(\frac{1}{2\sqrt{2}}\right)^2} \\ &= \frac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{8}} \\ &= \frac{\frac{1}{\sqrt{2}}}{\frac{7}{8}} \\ &= \frac{8}{7\sqrt{2}} \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right) \\ &= \frac{8\sqrt{2}}{14} = \frac{4\sqrt{2}}{7} \quad \left(\text{or } \frac{2\sqrt{8}}{7}\right)\end{aligned}$$

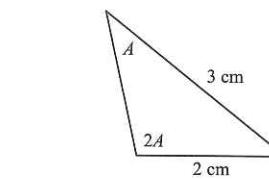
$$\begin{aligned}\text{11} \quad \text{a} \quad &(\sin\theta + \cos\theta)^2 \\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \\ &= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \\ &= 1 + \sin 2\theta\end{aligned}$$

$$\begin{aligned}\text{b} \quad &\cos^4\theta - \sin^4\theta \\ &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\ &= 1 \times \cos 2\theta \\ &= \cos 2\theta\end{aligned}$$



Using the sine rule,

$$\begin{aligned}\frac{\sin 2A}{7} &= \frac{\sin A}{5} \\ \therefore \frac{2\sin A \cos A}{7} &= \frac{\sin A}{5} \\ \therefore \cos A &= \frac{7}{10}\end{aligned}$$



$$\begin{aligned}\text{Using the sine rule,} \quad \frac{\sin 2A}{3} &= \frac{\sin A}{2} \\ \therefore \frac{2\sin A \cos A}{3} &= \frac{\sin A}{2} \\ \therefore \cos A &= \frac{3}{4}\end{aligned}$$

$$\text{13} \quad \text{a} \quad x \mapsto \tan x$$

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} \text{ is undefined when } \cos x = 0 \\ \therefore \text{when } x &= \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \\ \therefore \text{domain of } x \mapsto \tan x &= \{x \mid x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}\} \\ \tan x &\text{ can take any real value} \\ \therefore \text{range of } x \mapsto \tan x &= \{y \mid y \text{ is in } \mathbb{R}\}\end{aligned}$$

$$\text{b} \quad x \mapsto \sec 2x$$

$$\begin{aligned}\sec 2x &= \frac{1}{\cos 2x} \text{ is undefined when } \cos 2x = 0 \\ \therefore 2x &= \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \\ \therefore x &= \frac{\pi}{4} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}\end{aligned}$$

$$\therefore \text{domain of } x \mapsto \sec 2x = \{x \mid x \neq \frac{\pi}{4} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}\}$$

$$\text{Now } -1 \leqslant \cos 2x \leqslant 1$$

$$\therefore \frac{1}{\cos 2x} \leqslant -1 \text{ or } \frac{1}{\cos 2x} \geqslant 1$$

$$\therefore \text{range of } x \mapsto \sec 2x = \{y \mid y \geqslant 1 \text{ or } y \leqslant -1\}$$

$$\text{c} \quad x \mapsto \cot 3x$$

$$\begin{aligned}\cot 3x &= \frac{\cos 3x}{\sin 3x} \text{ is undefined when } \sin 3x = 0 \\ \therefore 3x &= k\pi, \quad k \in \mathbb{Z} \\ \therefore x &= \frac{k\pi}{3}, \quad k \in \mathbb{Z}\end{aligned}$$

$$\therefore \text{domain of } x \mapsto \cot 3x = \{x \mid x \neq \frac{k\pi}{3}, \quad k \in \mathbb{Z}\}$$

$$\cot 3x \text{ can take any value}$$

$$\therefore \text{range of } x \mapsto \cot 3x = \{y \mid y \text{ is in } \mathbb{R}\}$$

$$\text{14} \quad \text{a} \quad \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$\begin{aligned}&= \frac{1}{2} - \frac{1}{2}(1 - 2\sin^2\theta) \\ &= \frac{1}{2} - \frac{1}{2} + \sin^2\theta \\ &= \sin^2\theta \\ &= \cos^2\theta\end{aligned}$$

$$\begin{aligned}\text{b} \quad &\frac{1}{2} + \frac{1}{2}\cos 2\theta \\ &= \frac{1}{2} + \frac{1}{2}(2\cos^2\theta - 1) \\ &= \frac{1}{2} + \cos^2\theta - \frac{1}{2} \\ &= \cos^2\theta\end{aligned}$$

15 a $\frac{\sin 2\theta}{1 - \cos 2\theta} = \frac{2 \sin \theta \cos \theta}{1 - (1 - 2 \sin^2 \theta)}$
 $= \frac{2 \sin \theta \cos \theta}{1 - 1 + 2 \sin^2 \theta}$
 $= \frac{2 \sin \theta \cos \theta}{2 \sin^2 \theta}$
 $= \frac{\cos \theta}{\sin \theta}$
 $= \cot \theta$

b $\frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \frac{\sin \theta + 2 \sin \theta \cos \theta}{1 + \cos \theta + 2 \cos^2 \theta - 1}$
 $= \frac{\sin \theta(1 + 2 \cos \theta)}{\cos \theta(1 + 2 \cos \theta)}$
 $= \frac{\sin \theta}{\cos \theta}$
 $= \tan \theta$

16 $\sqrt{3} \sin x + \cos x = k \sin(x + b)$
 $= k[\sin x \cos b + \cos x \sin b]$
 $= k \cos b \sin x + k \sin b \cos x$

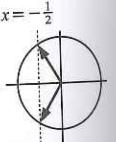
Equating coefficients of $\sin x$ and $\cos x$,
 $k \cos b = \sqrt{3}$ and $k \sin b = 1$ (1)
 $\therefore k^2 \cos^2 b = 3$ and $k^2 \sin^2 b = 1$ {squaring both sides}
 $\therefore k^2(\cos^2 b + \sin^2 b) = 4$ {adding the 2 equations}
 $\therefore k^2 = 4$
 $\therefore k = 2$ { $k > 0$ }

17 If $\sin A = \sin B$, then $\sin A - \sin B = 0$
 $\therefore 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B) = 0$

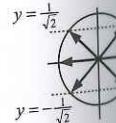
$\therefore \cos \frac{1}{2}(A+B) = 0$ or $\sin \frac{1}{2}(A-B) = 0$
 $\therefore \frac{1}{2}(A+B) = \frac{\pi}{2} + k\pi$ or $\frac{1}{2}(A-B) = k\pi$, $k \in \mathbb{Z}$
 $\therefore A+B = \pi + k2\pi$ or $A-B = k2\pi$, $k \in \mathbb{Z}$
 $\therefore A+B = \pi + k2\pi$ or $A = B + k2\pi$, $k \in \mathbb{Z}$

18 a $\cos 3\theta$
 $= \cos(2\theta + \theta)$
 $= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$
 $= (2 \cos^2 \theta - 1) \cos \theta - (2 \sin \theta \cos \theta) \sin \theta$
 $= 2 \cos^3 \theta - \cos \theta - 2 \sin^2 \theta \cos \theta$
 $= 2 \cos^3 \theta - \cos \theta - 2(1 - \cos^2 \theta) \cos \theta$
 $= 2 \cos^3 \theta - \cos \theta - 2 \cos \theta + 2 \cos^3 \theta$
 $= 4 \cos^3 \theta - 3 \cos \theta$

b $8 \cos^3 \theta - 6 \cos \theta + 1 = 0$
 $\therefore 8 \cos^3 \theta - 6 \cos \theta = -1$
 $\therefore 4 \cos^3 \theta - 3 \cos \theta = -\frac{1}{2}$
 $\therefore \cos 3\theta = -\frac{1}{2}$
 $\therefore 3\theta = -\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3},$
 $\quad \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$
 $\{3\theta \in [-3\pi, 3\pi]\}$
 $\therefore \theta = -\frac{8\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$



b $\sin 3\theta = \sin \theta$
 $\therefore -4 \sin^3 \theta + 3 \sin \theta = \sin \theta$
 $\therefore 4 \sin^3 \theta - 2 \sin \theta = 0$
 $\therefore 2 \sin \theta(2 \sin^2 \theta - 1) = 0$
 $\therefore \sin \theta = 0$ or $\sin^2 \theta = \frac{1}{2}$
 $\therefore \sin \theta = 0$ or $\sin \theta = \pm \frac{1}{\sqrt{2}}$
 $\therefore \theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi, \frac{9\pi}{4}, \frac{11\pi}{4}, 3\pi$



19 a $\sin 3\theta$
 $= \sin(2\theta + \theta)$
 $= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$
 $= (2 \sin \theta \cos \theta) \cos \theta + (1 - 2 \sin^2 \theta) \sin \theta$
 $= 2 \sin \theta \cos^2 \theta + \sin \theta - 2 \sin^3 \theta$
 $= 2 \sin \theta(1 - \sin^2 \theta) + \sin \theta - 2 \sin^3 \theta$
 $= 2 \sin \theta - 2 \sin^3 \theta + \sin \theta - 2 \sin^3 \theta$
 $= -4 \sin^3 \theta + 3 \sin \theta$

20 The period of a function $f(x)$ is the smallest $p > 0$ such that

$$\begin{aligned} f(x+p) &= f(x) \text{ for all } x \\ \therefore \sin[n(x+p)] &= \sin(nx) \\ \therefore \sin(nx+np) &= \sin(nx) \\ \therefore \sin(nx)\cos(np) + \cos(nx)\sin(np) &= \sin(nx) \end{aligned}$$

Equating coefficients of $\sin(nx)$ and $\cos(nx)$,

$\cos(np) = 1$ and $\sin(np) = 0$

$\therefore np = 2k\pi$, $k \in \mathbb{Z}$

$\therefore p = \frac{2k\pi}{n}$, $k \in \mathbb{Z}$

The smallest $p > 0$ occurs when $k = 1 \therefore p = \frac{2\pi}{n}$

21 a $2 \cos x - 5 \sin x = k \cos(x+b)$
 $= k[\cos x \cos b - \sin x \sin b]$
 $= k \cos b \cos x - k \sin b \sin x$

Equating coefficients of $\cos x$ and $\sin x$,

$k \cos b = 2$ and $k \sin b = 5$ (1)

$\therefore k^2 \cos^2 b = 4$ and $k^2 \sin^2 b = 25$ {squaring both sides}

$\therefore k^2(\cos^2 b + \sin^2 b) = 29$ {adding the two equations}

$\therefore k^2 = 29 \therefore k = \sqrt{29}$ { $k > 0$ }

Substituting $k = \sqrt{29}$ into (1) gives

$\cos b = \frac{2}{\sqrt{29}}$ and $\sin b = \frac{5}{\sqrt{29}}$

$\therefore b = \cos^{-1}\left(\frac{2}{\sqrt{29}}\right) \approx 1.19$

$\therefore 2 \cos x - 5 \sin x \approx \sqrt{29} \cos(x + 1.19)$

b $2 \cos x - 5 \sin x = -2$

$\therefore \sqrt{29} \cos(x + 1.19) \approx -2$

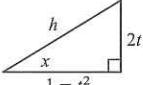
$\therefore \cos(x + 1.19) \approx -\frac{2}{\sqrt{29}}$

$\therefore x + 1.19 \approx 1.951, 4.33$

$\therefore x \approx 0.761, 3.14 \leftarrow$ this solution is exactly π

c $\tan x = \tan\left(2 \times \frac{x}{2}\right)$

$$\begin{aligned} &= \frac{2 \tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} \\ &= \frac{2t}{1 - t^2} \end{aligned}$$



Now $h^2 = (1-t^2)^2 + (2t)^2$

$= 1 - 2t^2 + t^4 + 4t^2$

$= 1 + 2t^2 + t^4$

$\therefore h = 1 + t^2$

$\therefore \sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

d $2 \cos x - 5 \sin x = -2$

$\therefore 2\left(\frac{1-t^2}{1+t^2}\right) - 5\left(\frac{2t}{1+t^2}\right) = -2$

$\therefore \frac{2-2t^2-10t}{1+t^2} = -2$

$\therefore 2-2t^2-10t = -2-2t^2$

$\therefore 4 = 10t$

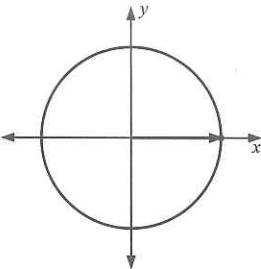
$\therefore t = \frac{2}{5}$

So $\tan\left(\frac{x}{2}\right) = \frac{2}{5}$

$\therefore \frac{x}{2} \approx 0.3805 \quad \left\{ 0 \leqslant \frac{x}{2} \leqslant \frac{\pi}{2} \right\}$

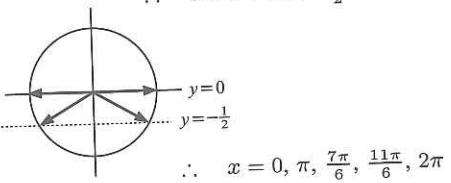
$\therefore x \approx 0.761$

The $x = \pi$ solution has been lost since t is undefined when $x = \pi$.

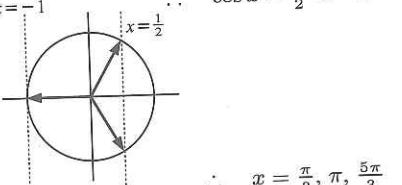


EXERCISE 12L

1 a $2 \sin^2 x + \sin x = 0$
 $\therefore \sin x(2 \sin x + 1) = 0$
 $\therefore \sin x = 0 \text{ or } -\frac{1}{2}$

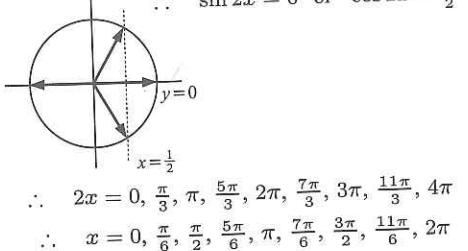


c $2 \cos^2 x + \cos x - 1 = 0$
 $\therefore (2 \cos x - 1)(\cos x + 1) = 0$
 $\therefore \cos x = \frac{1}{2} \text{ or } -1$

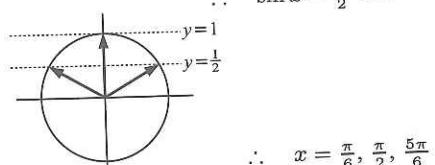


e $\sin^2 x = 2 - \cos x$
 $\therefore 1 - \cos^2 x = 2 - \cos x$
 $\therefore \cos^2 x - \cos x + 1 = 0$
where $\Delta = (-1)^2 - 4(1)(1)$
 $= 1 - 4$
 $= -3$
 $\therefore \text{no real solutions exist}$

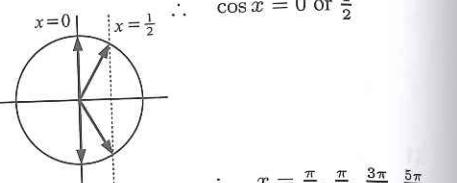
g $\sin 4x = \sin 2x$
 $\therefore 2 \sin 2x \cos 2x = \sin 2x$
 $\therefore \sin 2x(2 \cos 2x - 1) = 0$
 $\therefore \sin 2x = 0 \text{ or } \cos 2x = \frac{1}{2}$



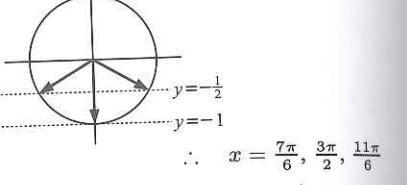
2 a $2 \sin x + \csc x = 3$
 $\therefore 2 \sin^2 x + 1 = 3 \sin x$
 $\therefore 2 \sin^2 x - 3 \sin x + 1 = 0$
 $\therefore (2 \sin x - 1)(\sin x - 1) = 0$
 $\therefore \sin x = \frac{1}{2} \text{ or } 1$



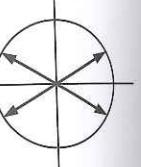
b $2 \cos^2 x = \cos x$
 $\therefore 2 \cos^2 x - \cos x = 0$
 $\therefore \cos x(2 \cos x - 1) = 0$
 $\therefore \cos x = 0 \text{ or } \frac{1}{2}$



d $2 \sin^2 x + 3 \sin x + 1 = 0$
 $\therefore (2 \sin x + 1)(\sin x + 1) = 0$
 $\therefore \sin x = -\frac{1}{2} \text{ or } -1$



f $3 \tan x = \cot x$
 $\therefore 3 \tan^2 x = 1$
 $\therefore \tan^2 x = \frac{1}{3}$
 $\therefore \tan x = \pm \frac{1}{\sqrt{3}}$
 $\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

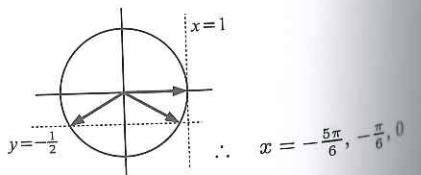


h $\sin x + \cos x = \sqrt{2}$
 $\therefore (\sin x + \cos x)^2 = 2$
 $\therefore \sin^2 x + 2 \sin x \cos x + \cos^2 x = 2$
 $\therefore 1 + \sin 2x = 2$
 $\therefore \sin 2x = 1$
 $\therefore 2x = \frac{\pi}{2}, \frac{5\pi}{2}$
 $\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

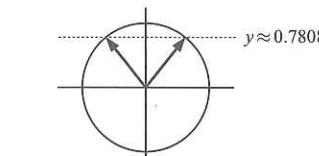
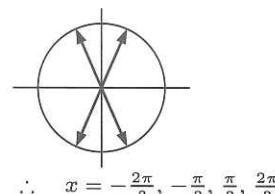
But when $x = \frac{5\pi}{4}$, $\sin x + \cos x = -\sqrt{2}$

$\therefore x = \frac{\pi}{4}$ is the only solution

b $\sin 2x + \cos x - 2 \sin x - 1 = 0$
 $\therefore 2 \sin x \cos x + \cos x - 2 \sin x - 1 = 0$
 $\therefore (2 \sin x + 1)(\cos x - 1) = 0$
 $\therefore \sin x = -\frac{1}{2} \text{ or } \cos x = 1$



c $\tan^4 x - 2 \tan^2 x - 3 = 0$
 $\therefore (\tan^2 x - 3)(\tan^2 x + 1) = 0$
 $\therefore \tan^2 x = 3 \text{ or } -1$
 $\therefore \tan^2 x = \pm \sqrt{3}$



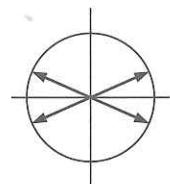
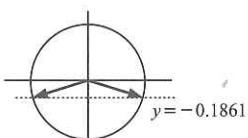
3 a $2 \cos^2 x = \sin x$
 $\therefore 2(1 - \sin^2 x) - \sin x = 0$
 $\therefore 2 - 2 \sin^2 x - \sin x = 0$
 $\therefore 2 \sin^2 x + \sin x - 2 = 0$

$$\therefore \sin x = \frac{-1 \pm \sqrt{1 - 4(2)(-2)}}{2(2)}$$

$$\therefore \sin x = \frac{-1 \pm \sqrt{17}}{4} \approx 0.7808 \text{ or } -1.281$$

b $\cos 2x + 5 \sin x = 0$
 $\therefore 1 - 2 \sin^2 x + 5 \sin x = 0$
 $\therefore 2 \sin^2 x - 5 \sin x - 1 = 0$
 $\therefore \sin x = \frac{5 \pm \sqrt{25 - 4(2)(-1)}}{2(2)}$

$$= \frac{5 \pm \sqrt{33}}{4} \approx 2.6861 \text{ or } -0.1861$$
 $\therefore \sin x \approx -0.1861 \quad \{ \text{as } -1 \leqslant \sin x \leqslant 1 \}$

**EXERCISE 12M**

1 a $1 + \sin x + \sin^2 x + \sin^3 x + \dots + \sin^{n-1} x$
is a geometric series with
 $u_1 = 1, r = \sin x$

$$\therefore \text{sum} = \frac{u_1(1 - r^n)}{1 - r}$$

$$= \frac{1(1 - \sin^n x)}{1 - \sin x}$$

$$= \frac{1 - \sin^n x}{1 - \sin x}$$

2 a i $2 \sin x(\cos x + \cos 3x)$
 $= 2 \sin x \cos x + 2 \sin x \cos 3x$
 $= \sin 2x + \sin 4x + \sin(-2x)$
 $\{ 2 \sin A \cos B = \sin(A + B) + \sin(A - B) \}$
 $= \sin 2x + \sin 4x - \sin 2x$
 $= \sin 4x$

b $S = \frac{u_1}{1 - r} = \frac{1}{1 - \sin x}$
as $-1 \leqslant \sin x \leqslant 1 \Rightarrow -1 \leqslant r \leqslant 1$

c If $S = \frac{2}{3}, \frac{1}{1 - \sin x} = \frac{2}{3}$
 $\therefore 3 = 2 - 2 \sin x$
 $\therefore 2 \sin x = -1$
 $\therefore \sin x = -\frac{1}{2}$
 $y = -\frac{1}{2}$
 $\therefore x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$

ii $2 \sin x(\cos x + \cos 3x + \cos 5x)$
 $= 2 \sin x(\cos x + \cos 3x) + 2 \sin x \cos 5x$
 $= \sin 4x + \sin 6x + \sin(-4x) \quad \{ \text{from i} \}$
 $= \sin 4x + \sin 6x - \sin 4x$
 $= \sin 6x$

- b**
- I $2 \sin x(\cos x + \cos 3x + \cos 5x + \dots + \cos 7x) = \sin 8x$
 - II $2 \sin x(\cos x + \cos 3x + \cos 5x + \dots + \cos 19x) = \sin 20x$
- $$\therefore \cos x + \cos 3x + \cos 5x + \dots + \cos 19x = \frac{\sin 20x}{2 \sin x}$$
- c** In general, $\cos x + \cos 3x + \cos 5x + \dots + \cos(2n-1)x = \frac{\sin 2nx}{2 \sin x}$

3 a i $\sin x \cos x \cos 2x$

$$\begin{aligned} &= \frac{1}{2}(2 \sin x \cos x) \cos 2x \\ &= \frac{1}{2} \sin 2x \cos 2x \\ &= \frac{1}{4} 2 \sin 2x \cos 2x \\ &= \frac{1}{4} \sin 4x \quad \dots \dots (1) \\ &= \frac{\sin(2^2 x)}{2^2} \end{aligned}$$

i $(\sin x \cos x \cos 2x) \cos 4x$

$$\begin{aligned} &= \frac{1}{4} \sin 4x \cos 4x \quad \{ \text{from (1)} \} \\ &= \frac{1}{8} (2 \sin 4x \cos 4x) \\ &= \frac{\sin 8x}{8} \\ &= \frac{\sin(2^3 x)}{2^3} \end{aligned}$$

b i $\frac{\sin(2^4 x)}{2^4}$

ii $\frac{\sin(2^6 x)}{2^6}$

c $\sin x \cos x \cos 2x \cos 4x \dots \cos(2^n x) = \frac{\sin(2^{n+1} x)}{2^{n+1}}$

or $\cos x \cos 2x \cos 4x \dots \cos(2^n x) = \frac{\sin(2^{n+1} x)}{2^{n+1} \sin x}$

- 4 a** P_n is “ $\cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2 \sin \theta}$ ”, $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

$$(1) \text{ If } n = 1, \text{ LHS} = \cos \theta \text{ and RHS} = \frac{\sin 2\theta}{2 \sin \theta} = \frac{2 \sin \theta \cos \theta}{2 \sin \theta} = \cos \theta \quad \therefore P_1 \text{ is true.}$$

(2) If P_k is true, then

$$\begin{aligned} \cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2k-1)\theta &= \frac{\sin 2k\theta}{2 \sin \theta} \\ \therefore \cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2k-1)\theta + \cos(2k+1)\theta &= \\ &= \frac{\sin 2k\theta + \cos(2k+1)\theta}{2 \sin \theta} \\ &= \frac{\sin 2k\theta + 2 \sin \theta \cos(2k+1)\theta}{2 \sin \theta} \\ &= \frac{\sin 2k\theta + \sin[\theta + (2k+1)\theta] + \sin[\theta - (2k+1)\theta]}{2 \sin \theta} \\ &= \frac{\sin 2k\theta + \sin(\theta + 2k\theta + \theta) + \sin(\theta - 2k\theta - \theta)}{2 \sin \theta} \\ &= \frac{\sin 2k\theta + \sin(2k\theta + 2\theta) + \sin(-2k\theta)}{2 \sin \theta} \\ &= \frac{\sin 2k\theta + \sin 2(k+1)\theta - \sin 2k\theta}{2 \sin \theta} \\ &= \frac{\sin 2(k+1)\theta}{2 \sin \theta} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

- b** $\cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos 31\theta = \frac{\sin 32\theta}{2 \sin \theta} \quad \{n = 16\}$

- 5** P_n is “ $\sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2n-1)\theta = \frac{1 - \cos 2n\theta}{2 \sin \theta}$ ”, $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

$$(1) \text{ If } n = 1, \text{ LHS} = \sin \theta \text{ and RHS} = \frac{1 - \cos 2\theta}{2 \sin \theta} = \frac{1 - (1 - 2 \sin^2 \theta)}{2 \sin \theta} = \frac{2 \sin^2 \theta}{2 \sin \theta} = \sin \theta$$

$$\therefore P_1 \text{ is true.}$$

(2) If P_k is true, then

$$\begin{aligned} \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2k-1)\theta &= \frac{1 - \cos 2k\theta}{2 \sin \theta} \\ \therefore \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2k-1)\theta + \sin(2k+1)\theta &= \\ &= \frac{1 - \cos 2k\theta + \sin(2k+1)\theta}{2 \sin \theta} \\ &= \frac{1 - \cos 2k\theta + 2 \sin(2k+1)\theta \sin \theta}{2 \sin \theta} \\ &= \frac{1 - \cos 2k\theta + \cos[(2k+1)\theta - \theta] - \cos[(2k+1)\theta + \theta]}{2 \sin \theta} \\ &= \frac{1 - \cos 2k\theta + \cos 2k\theta - \cos[(2k+2)\theta]}{2 \sin \theta} \\ &= \frac{1 - \cos 2(k+1)\theta}{2 \sin \theta} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

Thus $\sin \frac{\pi}{7} + \sin \frac{3\pi}{7} + \sin \frac{5\pi}{7} + \dots + \sin \frac{13\pi}{7}$ has $2n-1 = 13$ and $\theta = \frac{\pi}{7}$

$$\therefore n = 7 \text{ and } \theta = \frac{\pi}{7}$$

$$\therefore \text{the sum is } \frac{1 - \cos(2 \times 7 \times \frac{\pi}{7})}{2 \sin \frac{\pi}{7}} = \frac{1 - \cos 2\pi}{2 \sin \frac{\pi}{7}} = \frac{1 - 1}{2 \sin \frac{\pi}{7}} = 0$$

- 6** P_n is “ $\cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \dots \times \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \times \sin x}$ ”, $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

$$(1) \text{ If } n = 1, \text{ LHS} = \cos x, \text{ RHS} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x \quad \therefore P_1 \text{ is true.}$$

(2) If P_k is true, then

$$\begin{aligned} \cos x \times \cos 2x \times \cos 4x \times \dots \times \cos(2^{k-1}x) &= \frac{\sin(2^k x)}{2^k \sin x} \\ \therefore \cos x \times \cos 2x \times \cos 4x \times \dots \times \cos(2^{k-1}x) \times \cos(2^k x) &= \\ &= \frac{\sin(2^k x)}{2^k \sin x} \times \cos(2^k x) \\ &= \frac{2 \sin(2^k x) \cos(2^k x)}{2 \times 2^k \sin x} \\ &= \frac{\sin(2 \times 2^k x)}{2^{k+1} \sin x} \quad \{2 \sin \theta \cos \theta = \sin 2\theta\} \\ &= \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

7 P_n is “ $\cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2(n\theta) = \frac{1}{2} \left[n + \frac{\cos(n+1)\theta \sin n\theta}{\sin \theta} \right]$ ”, $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS = $\cos^2 \theta$, RHS = $\frac{1}{2} \left[1 + \frac{\cos 2\theta \sin \theta}{\sin \theta} \right]$
 $= \frac{1}{2} + \frac{1}{2}(2 \cos^2 \theta - 1)$
 $= \cos^2 \theta \quad \therefore P_1$ is true.

(2) If P_k is true then

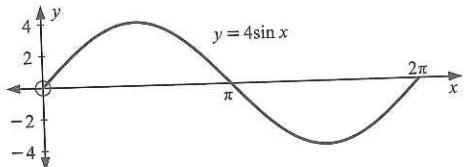
$$\begin{aligned} \cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2(k\theta) &= \frac{1}{2} \left[k + \frac{\cos(k+1)\theta \sin k\theta}{\sin \theta} \right] \\ \therefore \cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2(k\theta) + \cos^2(k+1)\theta &= \frac{1}{2} \left[k + \frac{\cos(k+1)\theta \sin k\theta}{\sin \theta} \right] + \cos^2(k+1)\theta \\ &= \frac{1}{2} \left[k + \frac{\cos(k+1)\theta \sin k\theta}{\sin \theta} \right] + \frac{1}{2} + \frac{1}{2} \cos 2(k+1)\theta \quad \{ \cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \} \\ &= \frac{1}{2}(k+1) + \frac{\cos(k+1)\theta \sin k\theta + \cos 2(k+1)\theta \sin \theta}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin[k\theta + (k+1)\theta] + \frac{1}{2} \sin[k\theta - (k+1)\theta] + \frac{1}{2} \sin[\theta + 2(k+1)\theta] + \frac{1}{2} \sin[\theta - 2(k+1)\theta]}{2 \sin \theta} \\ &\quad \{ \text{products to sums formula} \} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin(2k+1)\theta + \frac{1}{2} \sin(-\theta) + \frac{1}{2} \sin(2k+3)\theta + \frac{1}{2} \sin(-2k-1)\theta}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin(2k+1)\theta - \frac{1}{2} \sin \theta + \frac{1}{2} \sin(2k+3)\theta - \frac{1}{2} \sin(2k+1)\theta}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin(2k+3)\theta - \frac{1}{2} \sin \theta}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\cos \left[\frac{(2k+3)\theta + \theta}{2} \right] \sin \left[\frac{(2k+3)\theta - \theta}{2} \right]}{2 \sin \theta} \quad \{ \text{factor formula} \} \\ &= \frac{1}{2}(k+1) + \frac{\cos(k+2)\theta \sin(k+1)\theta}{2 \sin \theta} \\ &= \frac{1}{2} \left[(k+1) + \frac{\cos(k+2)\theta \sin(k+1)\theta}{\sin \theta} \right] \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

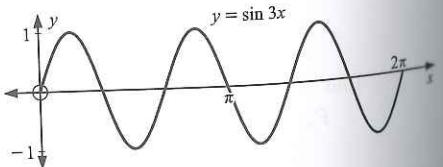
$\therefore P_n$ is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 12A

- 1 This is the graph of $y = \sin x$ under a vertical stretch of factor 4. The amplitude is 4.

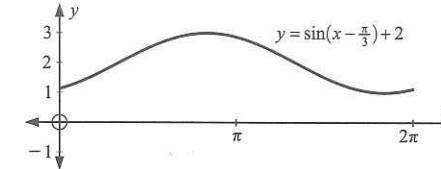


- 2 This is the graph of $y = \sin x$ under a horizontal compression of factor 3. The period is $\frac{2\pi}{3}$.



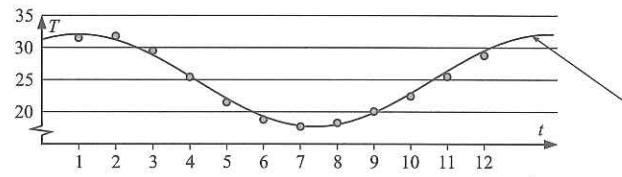
3 a period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$

b period = $\frac{\pi}{4}$



5

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8



$T = A \sin B(t - C) + D$ period = $\frac{2\pi}{B} = 12$, $\therefore B = \frac{2\pi}{12} = \frac{\pi}{6}$

max. = 31.8 $\therefore A = \frac{\text{max.} - \text{min.}}{2} \approx \frac{31.8 - 17.7}{2} \approx 7.05$

min. = 17.7 $D = \frac{\text{max.} + \text{min.}}{2} \approx \frac{31.8 + 17.7}{2} \approx 24.75$

$C = \frac{7 + 14}{2} = 10.5$ {values of t at min. and max.}

So, $T \approx 7.05 \sin \frac{\pi}{6}(t - 10.5) + 24.75$

From technology, $T \approx 7.21 \sin(0.488t + 1.082) + 24.75$

$\approx 7.21 \sin 0.488(t + 2.22) + 24.75$ Note: $2.22 - -10.5 \approx 12.7 \approx 12$

6 a $\sin x = 0.382$

$\therefore x \approx 0.392, 2.75, 6.68$

b $\tan \left(\frac{x}{2} \right) = -0.458$

$\therefore x \approx 5.42$

7 a $\sin(x - 2.4) = 0.754$

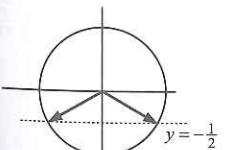
$\therefore x \approx 3.25, 4.69$

b $\sin \left(x + \frac{\pi}{3} \right) = 0.6049$

$\therefore x \approx 1.44, 5.89, 7.73$

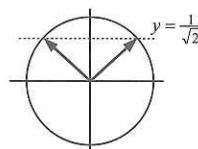
8 a $2 \sin x = -1$, $x \in [0, 4\pi]$

$\therefore \sin x = -\frac{1}{2}$



b $\sqrt{2} \sin x - 1 = 0$, $x \in [-2\pi, 2\pi]$

$\therefore \sin x = \frac{1}{\sqrt{2}}$



$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6} + k2\pi$

$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4} + k2\pi$

$\therefore x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

9 a $2 \sin 3x + \sqrt{3} = 0, x \in [0, 2\pi]$
 $\therefore \sin 3x = -\frac{\sqrt{3}}{2}$

$$\begin{aligned} 3x &= \left\{ \frac{4\pi}{3}, \frac{5\pi}{3} \right\} + k2\pi \\ \therefore x &= \left\{ \frac{4\pi}{9}, \frac{5\pi}{9} \right\} + k\frac{2\pi}{3} \\ \therefore x &= \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9} \end{aligned}$$

10 $P(t) = 5 + 2 \sin\left(\frac{\pi t}{3}\right), 0 \leq t \leq 8$, where $P(t)$ is in thousands of water beetles.

a $P(0) = 5 + 2 \sin 0 = 5$

So, 5000 water beetles.

b Smallest $P = 5 + 2(-1) = 3$

Largest $P = 5 + 2(1) = 7$

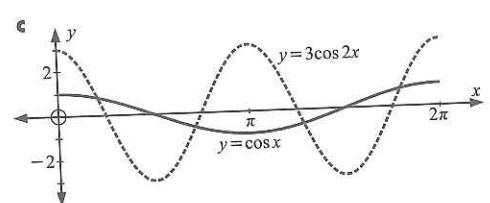
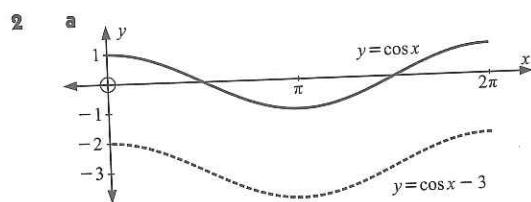
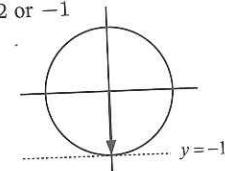
∴ smallest is 3000 water beetles
largest is 7000 water beetles

REVIEW SET 12B

1 a $\sin^2 x - \sin x - 2 = 0$
 $\therefore (\sin x - 2)(\sin x + 1) = 0$
 $\therefore \sin x = 2 \text{ or } -1$

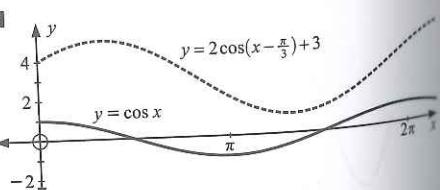
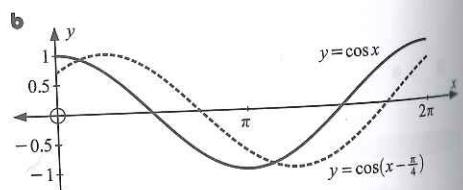
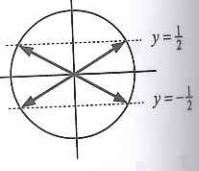
But $\sin x$ values lie between -1 and 1 inclusive

∴ $\sin x = -1$
 $\therefore x = \frac{3\pi}{2} + k2\pi$



b $4 \sin^2 x = 1$
 $\therefore \sin^2 x = \frac{1}{4}$
 $\therefore \sin x = \pm \frac{1}{2}$

∴ $x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k\pi$



3 $P(t) = 40 + 12 \sin \frac{2\pi}{7} \left(t - \frac{37}{12}\right) \text{ mg}$

a $P(t)$ has a minimum of $40 + 12(-1) = 28$ mg per m^3

b when $\sin \frac{2\pi}{7} \left(t - \frac{37}{12}\right) = -1$
 $\therefore \frac{2\pi}{7} \left(t - \frac{37}{12}\right) = \frac{3\pi}{2} + k2\pi$
 $\therefore \frac{2}{7} \left(t - \frac{37}{12}\right) = \frac{3}{2} + k2$

So, $t - \frac{37}{12} = \frac{21}{4} + k7$
 $\therefore t = 8\frac{1}{3} + k7$
 $\therefore t = 1\frac{1}{3}, 8\frac{1}{3}, 15\frac{1}{3}, \text{ and so on.}$

∴ on Mondays at 8.00 am
 $\{1\frac{1}{3} \text{ days after midnight Saturday}\}$

4 a If $y = A \cos B(t - C) + D$

then $A = -4, \frac{2\pi}{B} = \pi$
 $\therefore B = 2$

$C = D = 0$
 $\therefore y = -4 \cos 2x$

b If $y = A \cos B(x - C) + D$

then $A = 1, \frac{2\pi}{B} = 8 \therefore B = \frac{\pi}{4}$
 $D = \frac{\text{max.} + \text{min.}}{2} = \frac{3+1}{2} = 2$
 $C = 0$

So, $y = \cos\left(\frac{\pi}{4}x\right) + 2$

b $\cos(x - 2.4) = -0.6014, 0 \leq x \leq 6$
 $\therefore x \approx 0.184, 4.62$

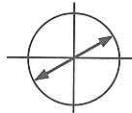
5 a $\cos x = 0.4379, 0 \leq x \leq 10$
 $\therefore x \approx 1.12, 5.17, 7.40$

6 a $\cos(165^\circ)$
 $= \cos(120^\circ + 45^\circ)$
 $= \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ$
 $= \left(-\frac{1}{2}\right) \left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right)$
 $= \frac{-1 - \sqrt{3}}{2\sqrt{2}}$
 $= \frac{-\sqrt{2} - \sqrt{6}}{4}$

b $\tan\left(\frac{\pi}{12}\right)$
 $= \tan\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right)$
 $= \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$
 $= \frac{1 - \frac{1}{\sqrt{3}}}{1 + (1)(\frac{1}{\sqrt{3}})} \times \left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{3 - 2\sqrt{3} + 1}{3 - 1}$
 $= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \left(\frac{\sqrt{3} - 1}{\sqrt{3} - 1}\right) = \frac{4 - 2\sqrt{3}}{2}$
 $= 2 - \sqrt{3}$

7 a $\tan(x - \frac{\pi}{3}) = \frac{1}{\sqrt{3}}, x \in [0, 4\pi]$

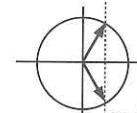
∴ $x - \frac{\pi}{3} = \frac{\pi}{6} + k\pi$
 $\therefore x = \frac{\pi}{2} + k\pi$



∴ $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

b $\cos(x + \frac{2\pi}{3}) = \frac{1}{2}, x \in [-2\pi, 2\pi]$

$x + \frac{2\pi}{3} = \left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\} + k2\pi$

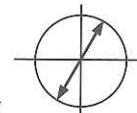


∴ $x = -\pi, -\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

8 a $\sqrt{2} \cos(x + \frac{\pi}{4}) - 1 = 0, x \in [0, 4\pi]$
 $\therefore \cos(x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

∴ $x + \frac{\pi}{4} = \left\{ \frac{\pi}{4}, \frac{7\pi}{4} \right\} + k2\pi$
 $\therefore x = \left\{ 0, \frac{3\pi}{2} \right\} + k2\pi$
 $\therefore x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi$

b $\tan 2x - \sqrt{3} = 0, x \in [0, 2\pi]$
 $\therefore \tan 2x = \sqrt{3}$



∴ $2x = \frac{\pi}{3} + k\pi$
 $\therefore x = \frac{\pi}{6} + k\frac{\pi}{2}$
 $\therefore x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

$$\begin{aligned} 9 \quad a \quad & \cos^3 \theta + \sin^2 \theta \cos \theta \\ &= \cos \theta (\cos^2 \theta + \sin^2 \theta) \\ &= \cos \theta (1) \\ &= \cos \theta \end{aligned}$$

$$\begin{aligned} b \quad & \frac{\cos^2 \theta - 1}{\sin \theta} \\ &= \frac{-(1 - \cos^2 \theta)}{\sin \theta} \\ &= -\frac{\sin^2 \theta}{\sin \theta} \\ &= -\sin \theta \end{aligned}$$

$$\begin{aligned} c \quad & 5 - 5 \sin^2 \theta \\ &= 5(1 - \sin^2 \theta) \\ &= 5 \cos^2 \theta \end{aligned}$$

$$\begin{aligned} d \quad & \frac{\sin^2 \theta - 1}{\cos \theta} \\ &= -\frac{(1 - \sin^2 \theta)}{\cos \theta} \\ &= -\frac{\cos^2 \theta}{\cos \theta} \\ &= -\cos \theta \end{aligned}$$

$$\begin{aligned} e \quad & \cos^2 \theta (\tan \theta + 1)^2 - 1 \\ &= \cos^2 \theta (\tan^2 \theta + 2 \tan \theta + 1) - 1 \\ &= \cos^2 \theta (\sec^2 \theta + 2 \tan \theta) - 1 \\ &= 1 + 2 \cos^2 \theta \left(\frac{\sin \theta}{\cos \theta} \right) - 1 \\ &= 2 \sin \theta \cos \theta \\ &= \sin 2\theta \end{aligned}$$

$$\begin{aligned} 10 \quad a \quad & (2 \sin \alpha - 1)^2 \\ &= 4 \sin^2 \alpha - 4 \sin \alpha + 1 \end{aligned}$$

$$\begin{aligned} b \quad & (\cos \alpha - \sin \alpha)^2 \\ &= \cos^2 \alpha - 2 \sin \alpha \cos \alpha + \sin^2 \alpha \\ &= \cos^2 \alpha + \sin^2 \alpha - 2 \sin \alpha \cos \alpha \\ &= 1 - \sin 2\alpha \end{aligned}$$

REVIEW SET 12C

$$\begin{aligned} 1 \quad a \quad & \frac{1 - \cos^2 \theta}{1 + \cos \theta} \\ &= \frac{(1 + \cos \theta)(1 - \cos \theta)}{1 + \cos \theta} \\ &= 1 - \cos \theta \end{aligned}$$

$$\begin{aligned} b \quad & \frac{\sin \alpha - \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\ &= \frac{1}{(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)} \\ &= \frac{1}{\sin \alpha + \cos \alpha} \end{aligned}$$

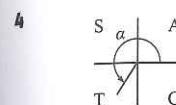
$$\begin{aligned} c \quad & \frac{4 \sin^2 \alpha - 4}{8 \cos \alpha} \\ &= \frac{-4(1 - \sin^2 \alpha)}{8 \cos \alpha} \\ &= \frac{-4 \cos^2 \alpha}{8 \cos \alpha} \\ &= -\frac{1}{2} \cos \alpha \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad & \frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} \\ &= \frac{\cos^2 \theta + (1 + \sin \theta)^2}{(1 + \sin \theta) \cos \theta} \\ &= \frac{\cos^2 \theta + 1 + 2 \sin \theta + \sin^2 \theta}{(1 + \sin \theta) \cos \theta} \\ &= \frac{2 + 2 \sin \theta}{(1 + \sin \theta) \cos \theta} \quad \{ \cos^2 \theta + \sin^2 \theta = 1 \} \\ &= \frac{2(1 + \sin \theta)}{(1 + \sin \theta) \cos \theta} \\ &= \frac{2}{\cos \theta} = 2 \sec \theta \end{aligned}$$

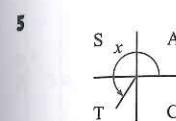
$$\begin{aligned} 3 \quad a \quad & \sin 2A = 2 \sin A \cos A \\ &= 2 \left(\frac{5}{13} \right) \left(\frac{12}{13} \right) \\ &= \frac{120}{169} \end{aligned}$$

$$\begin{aligned} b \quad & \left(1 + \frac{1}{\cos \theta} \right) (\cos \theta - \cos^2 \theta) \\ &= \cancel{\cos \theta} - \cos^2 \theta + 1 - \cancel{\cos \theta} \\ &= 1 - \cos^2 \theta \\ &= \sin^2 \theta \end{aligned}$$

$$\begin{aligned} b \quad & \cos 2A = \cos^2 A - \sin^2 A \\ &= \left(\frac{12}{13} \right)^2 - \left(\frac{5}{13} \right)^2 \\ &= \frac{144 - 25}{169} \\ &= \frac{119}{169} \end{aligned}$$



$$\begin{aligned} \cos^2 \alpha + \sin^2 \alpha &= 1 \\ \therefore \cos^2 \alpha + \frac{9}{16} &= 1 \\ \therefore \cos^2 \alpha &= \frac{7}{16} \\ \therefore \cos \alpha &= \pm \frac{\sqrt{7}}{4} \\ \text{But in Q3, } \cos \alpha &< 0 \\ \therefore \cos \alpha &= -\frac{\sqrt{7}}{4} \end{aligned}$$



$$\begin{aligned} \cos 2A &= 1 - 2 \sin^2 A \\ \therefore \cos x &= 1 - 2 \sin^2 \left(\frac{x}{2} \right) \quad \{ \text{letting } 2A = x, A = \frac{x}{2} \} \\ \therefore -\frac{3}{4} &= 1 - 2 \sin^2 \left(\frac{x}{2} \right) \\ \therefore 2 \sin^2 \left(\frac{x}{2} \right) &= \frac{7}{4} \\ \therefore \sin^2 \left(\frac{x}{2} \right) &= \frac{7}{8} \\ \therefore \sin \left(\frac{x}{2} \right) &= \pm \frac{\sqrt{7}}{2\sqrt{2}} \\ \text{But } \frac{\pi}{2} &< \frac{x}{2} < \frac{3\pi}{4} \quad (\text{in Q2}) \quad \therefore \sin \left(\frac{x}{2} \right) = \frac{\sqrt{7}}{2\sqrt{2}} \end{aligned}$$

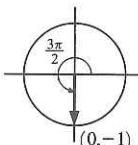
$$\begin{aligned} 6 \quad a \quad i \quad & \tan x = 4 \\ & \therefore x \approx 1.326 + k\pi \\ & \therefore x \approx 1.33 + k\pi \end{aligned} \quad \begin{aligned} ii \quad & \tan \left(\frac{x}{4} \right) = 4 \\ & \therefore \frac{x}{4} \approx 1.326 + k\pi \\ & \therefore x \approx 5.30 + k4\pi \end{aligned} \quad \begin{aligned} iii \quad & \tan(x - 1.5) = 4 \\ & \therefore x - 1.5 \approx 1.326 + k\pi \\ & \therefore x \approx 2.83 + k\pi \end{aligned}$$

$$\begin{aligned} b \quad i \quad & \tan(x + \frac{\pi}{6}) = -\sqrt{3} \\ & \therefore x + \frac{\pi}{6} = \frac{2\pi}{3} + k\pi \\ & \therefore x = \frac{\pi}{2} + k\pi \end{aligned} \quad \begin{aligned} ii \quad & \tan 2x = -\sqrt{3} \\ & \therefore 2x = \frac{2\pi}{3} + k\pi \\ & \therefore x = \frac{\pi}{3} + \frac{k\pi}{2} \end{aligned} \quad \begin{aligned} iii \quad & \tan^2 x - 3 = 0 \\ & \therefore \tan x = \pm \sqrt{3} \\ & \therefore x = \frac{\pi}{3} + k\pi \end{aligned}$$

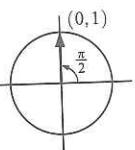
$$\begin{aligned} 7 \quad & \tan \theta = -\frac{2}{3}, \quad \frac{\pi}{2} < \theta < \pi \\ & \frac{\sin \theta}{\cos \theta} = -\frac{2}{3} \\ & \therefore \sin \theta = -2k, \quad \cos \theta = 3k \\ & \text{but } \cos^2 \theta + \sin^2 \theta = 1 \\ & \therefore 9k^2 + 4k^2 = 1 \\ & \therefore 13k^2 = 1 \\ & \therefore k = \pm \frac{1}{\sqrt{13}} \end{aligned} \quad \begin{aligned} \text{But in Q2, } \sin \theta &> 0, \quad \cos \theta < 0 \\ \therefore k &= -\frac{1}{\sqrt{13}} \\ \therefore \sin \theta &= \frac{2}{\sqrt{13}}, \quad \cos \theta = -\frac{3}{\sqrt{13}} \end{aligned}$$

$$\begin{aligned} 8 \quad & \frac{\sin 2\alpha - \sin \alpha}{\cos 2\alpha - \cos \alpha + 1} = \frac{2 \sin \alpha \cos \alpha - \sin \alpha}{2 \cos^2 \alpha - 1 - \cos \alpha + 1} \\ &= \frac{\sin \alpha (2 \cos \alpha - 1)}{\cos \alpha (2 \cos \alpha - 1)} = \frac{\sin \alpha}{\cos \alpha} = \tan \alpha \end{aligned}$$

$$\begin{aligned} 9 \quad a \quad & \cos \left(\frac{3\pi}{2} - \theta \right) \\ &= \cos \left(\frac{3\pi}{2} \right) \cos \theta + \sin \left(\frac{3\pi}{2} \right) \sin \theta \\ &= (0) \cos \theta + (-1) \sin \theta \\ &= -\sin \theta \end{aligned}$$

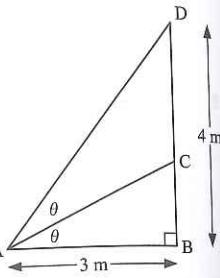


$$\begin{aligned} \mathbf{b} \quad & \sin\left(\theta + \frac{\pi}{2}\right) \\ &= \sin\theta \cos\left(\frac{\pi}{2}\right) + \cos\theta \sin\left(\frac{\pi}{2}\right) \\ &= \sin\theta(0) + \cos\theta(1) \\ &= \cos\theta \end{aligned}$$

10 Let $BC = x$ m

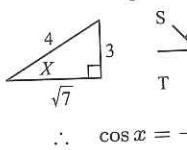
$$\begin{aligned} \therefore \tan\theta &= \frac{x}{3}, \quad \tan 2\theta = \frac{4}{3} \\ \therefore \frac{2\tan\theta}{1 - \tan^2\theta} &= \frac{4}{3} \\ \therefore 6\tan\theta &= 4 - 4\tan^2\theta \\ \therefore 2(2\tan^2\theta + 3\tan\theta - 2) &= 0 \\ \therefore 2(2\tan\theta - 1)(\tan\theta + 2) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \tan\theta &= \frac{1}{2} \text{ or } -2 \\ \text{But } \theta \text{ is clearly acute,} \\ \text{so } \tan\theta &> 0 \\ \therefore \tan\theta &= \frac{1}{2} \\ \therefore \frac{x}{3} &= \frac{1}{2} \\ \therefore x &= 1.5 \\ \therefore BC &= 1.5 \text{ m} \end{aligned}$$

**REVIEW SET 12D**

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right) \\ &= \sqrt{2} \left[\cos\theta \cos\left(\frac{\pi}{4}\right) - \sin\theta \sin\left(\frac{\pi}{4}\right) \right] \\ &= \sqrt{2} \left[\cos\theta \times \frac{1}{\sqrt{2}} - \sin\theta \times \frac{1}{\sqrt{2}} \right] \\ &= \cos\theta - \sin\theta \end{aligned}$$

2 a



$$\begin{aligned} \mathbf{b} \quad & \sin 2x \\ &= 2 \sin x \cos x \\ &= 2 \left(\frac{3}{4} \right) \left(-\frac{\sqrt{7}}{4} \right) \\ &= -\frac{3\sqrt{7}}{8} \end{aligned}$$

3

$$\begin{aligned} \cos 2\theta &= 1 - 2\sin^2\theta \\ \therefore \cos\left(\frac{\pi}{4}\right) &= 1 - 2\sin^2\left(\frac{\pi}{8}\right) \quad \{\text{letting } \theta = \frac{\pi}{8}\} \\ \therefore \frac{1}{\sqrt{2}} &= 1 - 2\sin^2\left(\frac{\pi}{8}\right) \\ \therefore 2\sin^2\left(\frac{\pi}{8}\right) &= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} \\ \therefore \sin^2\left(\frac{\pi}{8}\right) &= \frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ \therefore \sin^2\left(\frac{\pi}{8}\right) &= \frac{2-\sqrt{2}}{4} \\ \therefore \sin\left(\frac{\pi}{8}\right) &= \pm \frac{\sqrt{2-\sqrt{2}}}{2} \end{aligned}$$

But $\sin\left(\frac{\pi}{8}\right)$ is positive as $\frac{\pi}{8}$ is in quadrant 1.

$$\therefore \sin\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & (\sin\theta + \cos\theta)^2 \\ &= \sin^2\theta + 2\sin\theta \cos\theta + \cos^2\theta \\ &= [\sin^2\theta + \cos^2\theta] + 2\sin\theta \cos\theta \\ &= 1 + \sin 2\theta \end{aligned}$$

c

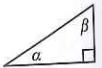
$$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x \\ &= 1 - 2\left(\frac{9}{16}\right) \\ &= 1 - \frac{9}{8} \\ &= -\frac{1}{8} \end{aligned}$$

d

$$\begin{aligned} \tan 2x &= \frac{\sin 2x}{\cos 2x} \\ &= \frac{-\frac{3\sqrt{7}}{8}}{-\frac{1}{8}} \\ &= 3\sqrt{7} \end{aligned}$$

$$\mathbf{4} \quad \alpha + \beta = \frac{\pi}{2} \quad \{\text{angles of a } \Delta\}$$

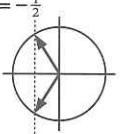
$$\therefore \beta = \frac{\pi}{2} - \alpha$$



$$\begin{aligned} \text{So, } \sin 2\beta &= \sin(\pi - 2\alpha) \\ &= \sin\pi \cos 2\alpha - \cos\pi \sin 2\alpha \\ &= (0)\cos 2\alpha - (-1)\sin 2\alpha \\ &= \sin 2\alpha \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \csc(2x) + \cot(2x) \\ &= \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x} \\ &= \frac{1 + \cos 2x}{\sin 2x} \\ &= \cot x \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad & 2\cos(2x) + 1 = 0 \\ \therefore \cos(2x) &= -\frac{1}{2} \\ \therefore 2x &= \frac{2\pi}{3} \cup \frac{4\pi}{3} + k2\pi \\ \therefore x &= \frac{\pi}{3} \cup \frac{2\pi}{3} + k\pi \\ \therefore x &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \end{aligned}$$



$$\mathbf{7} \quad \mathbf{a} \quad \sin 2\theta = 2\sin\theta \cos\theta$$

$$\begin{aligned} &= 2\left(\frac{b}{c}\right)\left(\frac{a}{c}\right) \\ &= \frac{2ab}{c^2} \end{aligned}$$

$$\mathbf{8} \quad \tan 2\alpha = \frac{4}{3}, \quad 0 < \alpha < \frac{\pi}{2}$$

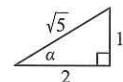
$$\begin{aligned} \therefore \frac{2\tan\alpha}{1 - \tan^2\alpha} &= \frac{4}{3} \\ \therefore 6\tan\alpha &= 4 - 4\tan^2\alpha \\ \therefore 4\tan^2\alpha + 6\tan\alpha - 4 &= 0 \\ \therefore 2(2\tan^2\alpha + 3\tan\alpha - 2) &= 0 \\ \therefore 2(2\tan\alpha - 1)(\tan\alpha + 2) &= 0 \end{aligned}$$

$$\therefore \tan\alpha = \frac{1}{2} \text{ or } -2$$

$$\begin{aligned} \mathbf{b} \quad & \sin 2x = -\sqrt{3} \cos 2x \\ \therefore \frac{\sin 2x}{\cos 2x} &= -\sqrt{3} \\ \therefore \tan 2x &= -\sqrt{3} \\ \therefore 2x &= \frac{2\pi}{3} + k\pi \\ \therefore x &= \frac{\pi}{3} + \frac{k\pi}{2} \\ \therefore x &= \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6} \end{aligned}$$

But α is in quadrant 1.

$$\therefore \tan\alpha = \frac{1}{2}$$



$$\therefore \sin\alpha = \frac{1}{\sqrt{5}}$$

$$\mathbf{b} \quad \text{Using the cosine rule:}$$

$$\begin{aligned} 3^2 &= x^2 + 5^2 - 2 \times x \times 5 \times \cos\alpha \\ \therefore 9 &= x^2 + 25 - 10x\left(\frac{5}{6}\right) \\ \therefore x^2 - \frac{25}{3}x + 16 &= 0 \\ \therefore 3x^2 - 25x + 48 &= 0 \end{aligned}$$

$$\mathbf{c} \quad (3x - 16)(x - 3) = 0$$

$$\therefore x = \frac{16}{3} \text{ or } 3$$

10 Let the shooter be x m from the wall.

$$\begin{aligned} \therefore \tan\alpha &= \frac{20}{x}, \quad \tan 2\alpha = \frac{45}{x} \\ \therefore \frac{2\tan\alpha}{1 - \tan^2\alpha} &= \frac{45}{x} \\ \therefore 2x\tan\alpha &= 45 - 45\tan^2\alpha \\ \therefore 2x\left(\frac{20}{x}\right) &= 45 - 45\left(\frac{20}{x}\right)^2 \\ \therefore 40 &= 45 - \frac{18000}{x^2} \\ \therefore \frac{18000}{x^2} &= 5 \\ \therefore x^2 &= 3600 \\ \therefore x &= 60 \end{aligned}$$

So, the shooter is 60 m from the wall.

